ΑΡΙΣΤΟΤΕΛΕΙΟ
ΠΑΝΕΠΙΣΤΗΜΙΟ
ΘΕΣΣΑΛΟΝΙΚΗΣ

ΑΝΟΙΚΤΑ
ΑΚΑΔΗΜΑΙΚΑ
ΜΑΘΗΜΑΤΑ

Αναπαραστάσεις Αλγεβρών Lie

Ενότητα 1: Αναπαραστάσεις Αλγεβρών Lie

Κ. Δασκαλογιάννης
Τμήμα Μαθηματικών

Α.Π.Θ.

Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο
ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ
επένδυση στην κοινωνία της γνώσης
ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ ΚΑΙ ΘΡΗΣΚΕΥΜΑΤΩΝ
ΕΙΔΙΚΗ ΥΠΗΡΕΣΙΑ ΔΙΑΧΕΙΡΙΣΗΣ
ΕΣΠΑ 2007-2013
πρόγραμμα για την ανάπτυξη
ΕΥΡΩΠΑΪΚΟ ΚΟΙΝΩΝΙΚΟ ΤΑΜΕΙΟ
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

Άδειες Χρήσης

Χρηματοδότηση

- Το παρόν εκπαιδευτικό υλικό έχει αναπτυχθεί στα πλαίσια του εκπαιδευτικού έργου του διδάσκοντα.
- Το έργο 'Ανοικτά Ακαδημαικά Μαθήματα στο Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης' έχει χρηματοδοτήσει μόνο την αναδιαμόρφωση του εκπαιδευτικού υλικού.
- Το έργο υλοποιείται στο πλαίσιο του Επιχειρησιακού Προγράμματος 'Εκπαίδευση και Δια Βίου Μάθηση' και συγχρηματοδοτείται απο την Ευρωπαική Ένωση (Ευρωπαικό Κοινωνικό Ταμείο) και από εθνικούς πόρους.

Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

Περιεχόμενα Ενότητας

- Εισαγωγή.
- Απλές και ημίαπλες Lie άλγεβρες.
- Περιβάλλουσες άλγεβρες.
- Δομή Hopf.
- Αναπαραστάσεις και modules.
- Παραδείγματα εφαρμογών.

Σκοποί Ενότητας

- Εισαγωγή των μεταπτυχιακών φοιτητών στην μελέτη και στις τεχνικές αλγεβρών μη προσεταιριστικών όπως είναι οι Lie άλγεβρες.

Definition of an Algebra

$\mathbb{F}$ = Field of characteristic 0 (ex $\mathbb{R}$, $\mathbb{C}$),
$\alpha, \beta, \ldots \in \mathbb{F}$ and $x, y, \ldots \in \mathcal{A}$

Definition of an Algebra $\mathcal{A}$

$\mathcal{A}$ is a $\mathbb{F}$-vector space with an additional distributive binary operation or product

$$\mathcal{A} \times \mathcal{A} \ni (x, y) \xrightarrow{m} m(x, y) = x * y \in \mathcal{A}$$

- $\mathbb{F}$-vector space
  $\alpha x = x\alpha,\ (\alpha + \beta)x = \alpha x + \beta x,\ \alpha(x + y) = \alpha x + \alpha y$
- distributivity
  $(x + y) * z = x * z + y * z,\ x * (y + z) = x * y + x * z$
  $\alpha(x * y) = (\alpha x) * y = x * (\alpha y) = (x * y)\alpha$

associative algebra $\equiv (x * y) * z = x * (y * z)$,
NON associative algebra $\equiv (x * y) * z \neq x * (y * z)$
commutative algebra $\equiv x * y = y * x$,
NON commutative algebra $\equiv x * y \neq y * x$

Definitions

$\mathbb{F}$ = Field of characteristic 0 (ex $\mathbb{R}$, $\mathbb{C}$),
$\mathfrak{g}$ = $\mathbb{F}$-Algebra with a product or **bracket** or **commutator**
$\alpha, \beta, \ldots \in \mathbb{F}$ and $x, y, \ldots \in \mathfrak{g}$

$$\mathfrak{g} \times \mathfrak{g} \ni (x, y) \longrightarrow [x, y] \in \mathfrak{g}$$

Definition :Lie algebra Axioms

(L-i) **bi-linearity**:
$$[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z]$$
$$[x, \alpha y + \beta z] = \alpha [x, y] + \beta [x, z]$$

(L-ii) **anti-commutativity**: $[x, y] = -[y, x] \Leftrightarrow [x, x] = 0$

(L-iii) **Jacobi identity**: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
or **Leibnitz rule**: $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

Examples of Lie algebras (1)

(i) $\mathfrak{g} = \{\vec{x}, \vec{y}, \ldots\}$ real vector space $\mathbb{R}^3$

$$[\vec{x},\, \vec{y}] \equiv \vec{x} \times \vec{y}$$

(ii) $\mathcal{A}$ associative $\mathbb{F}$- algebra $\rightsquigarrow$ $\mathcal{A}$ a Lie algebra with commutator $[A,\, B] = AB - BA$

(iii) Angular Momentum in Quantum Mechanics

$$L_1,\ L_2,\ L_3 \qquad \left\{ \begin{array}{c} \text{operators on} \\ \text{analytic (holomorphic)} \\ \text{functions} \end{array} \right\} f(x, y, z)$$

$$L_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad L_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad L_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\mathfrak{g} = \mathrm{span}\,\{L_1, L_2, L_3\}$$

$$[L_i,\, L_j] \equiv L_i \circ L_j - L_j \circ L_i$$

$$\rightsquigarrow [L_1,\, L_2] = -L_3, \quad [L_2,\, L_3] = -L_1, \quad [L_3,\, L_1] = -L_2$$

Examples of Lie algebras (2)

(iv) $V = \mathbb{F}$-vector space, $\dim V = n < \infty$
b bilinear form on V

$$b: \; V \times V \ni (v, w) \longrightarrow b(u, v) \in \mathbb{F}$$

$$b(\alpha u + \beta v, w) = \alpha b(u, w) + \beta b(v, w)$$
$$b(u, \alpha v + \beta w) = \alpha b(u, v) + \beta b(u, w)$$

$\mathfrak{o}(V, b) =$ *Orthogonal Lie algebra* $\equiv$ the set of all $T \in \mathrm{End}\, V$

$$b(u, Tv) + b(Tu, v) = 0$$

$$[T_1, \; T_2] = T_1 \circ T_2 - T_2 \circ T_1$$

$$T_1 \text{ and } T_2 \; \in \mathfrak{o}(V, b) \; \rightsquigarrow \; [T_1, \; T_2] \in \mathfrak{o}(V, b)$$

Matrix Formulation for bilinear forms

$$V \ni a \longleftrightarrow \bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \langle \bar{a}, \bar{b} \rangle = \sum_{i=1}^{n} a_i b_i = (a_1, a_2, \ldots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\mathrm{End}(V) \ni M \longleftrightarrow \mathbb{M} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ & \cdots & \cdots & \\ & \cdots & \cdots & \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{pmatrix} \rightsquigarrow \langle \mathbb{M}\bar{a}, \bar{b} \rangle = \langle \bar{a}, \mathbb{M}^{\mathrm{tr}} \bar{b} \rangle$$

bilinear form $b \longleftrightarrow \mathbb{B}$ matrix $n \times n \rightsquigarrow b(x, y) = \langle \bar{x}, \mathbb{B}\bar{y} \rangle$

$M \in O(V, b)$
$b(Mx, My) = b(x, y) \leftrightsquigarrow \mathbb{M}^{\mathrm{tr}}\mathbb{B}\mathbb{M} = \mathbb{B} \leftrightsquigarrow M^{\mathrm{tr}} b M = b$

$T \in \mathfrak{o}(V, b)$
$b(Tx, y) + b(x, Ty) = 0 \leftrightsquigarrow \mathbb{T}^{\mathrm{tr}}\mathbb{B} + \mathbb{B}\mathbb{T} = 0_{n \times n} \leftrightsquigarrow T^{\mathrm{tr}} b + bT = 0$

(v) A **Poisson algebra** is a vector space over a field $\mathbb{F}$ equipped with two bilinear products, $\cdot$ and $\{\,,\,\}$, having the following properties:

(a) The product $\cdot$ forms an associative (commutative) $\mathbb{F}$-algebra.

(b) The product $\{\,,\,\}$, called the **Poisson bracket**, forms a Lie algebra, and so it is anti-symmetric, and obeys the Jacobi identity.

(c) The Poisson bracket acts as a derivation of the associative product $\cdot$, so that for any three elements x, y and z in the algebra, one has $\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$

Example: **Poisson algebra on a Poisson manifold**

M is a manifold, $C^\infty(M)$ the "smooth"/analytic (complex) functions on the manifold.

$$\{f, g\} = \sum_{ij} \omega_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial j}{\partial x_g}$$

$$\omega_{ij}(x) = -\omega_{ji}(x), \quad \sum_m \left(\omega_{km} \frac{\partial \omega_{ij}}{\partial x_m} + \omega_{im} \frac{\partial \omega_{jk}}{\partial x_m} + \omega_{jm} \frac{\partial \omega_{ki}}{\partial x_m} \right) = 0$$

Structure Constants

$$\mathfrak{g} = \operatorname{span}(e_1,\, e_2,\, \ldots,\, e_n) = \mathbb{F}e_1 + \mathbb{F}e_2 + \cdots + \mathbb{F}e_n$$

$$[e_i,\, e_j] = \sum_{k=1}^{n} c_{ij}^{k}\, e_k \equiv c_{ij}^{k}\, e_k,\; c_{ij}^{k} \longleftrightarrow \textbf{structure constants}$$

(L-i) **bi-linearity**

(L-ii) **anti-commutativity**: $c_{ij}^{k} = -c_{ji}^{k}$

(L-iii) **Jacobi identity**: $c_{i\,j}^{m} c_{m\,k}^{\ell} + c_{j\,k}^{m} c_{m\,i}^{\ell} + c_{k\,i}^{m} c_{m\,j}^{\ell} = 0$

summation on **color** indices

$\mathfrak{gl}(V)$ algebra

$V = \mathbb{F}$-vector space, $\dim V = n < \infty$

general linear algebra $\mathfrak{gl}(V)$

$\mathfrak{gl}(V) \equiv \mathrm{End}(V) =$ endomorphisms on V $\dim \mathrm{gl}(V) = n^2$
$\mathfrak{gl}(V)$ is a Lie algebra with commutator $[A, B] = AB - BA$

$\mathfrak{gl}(\mathbb{C}^n) \equiv \mathfrak{gl}(n, \mathbb{C}) \equiv \mathbf{M}_n(\mathbb{C}) \equiv$ complex $n \times n$ matrices

Standard basis

Standard basis for $\mathfrak{gl}(n, \mathbb{C}) =$ matrices E_{ij} (1 in the (i, j) position, 0 elsewhere) $[E_{ij}, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}$

General Linear Group

$\mathrm{GL}(n,\mathbb{F})$ =invertible $n \times n$ matrices
$M \in \mathrm{GL}(n,\mathbb{F}) \leftrightsquigarrow \det M \neq 0$

$\mathrm{GL}(n,\mathbb{C})$ is an analytic manifold of dim n^2 AND a group.

The group multiplication is a continuous application $G \times G \longrightarrow G$
The group inversion is a continuous application $G \longrightarrow G$

Definition of the Lie algebra from the Lie group

$$\mathfrak{g}(n,\mathbb{C}) = T_{\mathbb{I}}\left(\mathrm{GL}(n,\mathbb{C})\right)$$

$$\left.\begin{array}{c} X(t) \in \mathrm{GL}(n,\mathbb{C}) \\ X(t) \text{ smooth trajectory} \\ X(0) = \mathbb{I} \end{array}\right\} \Longrightarrow \left\{X'(0) = x \in \mathfrak{gl}(n,\mathbb{C})\right\}$$

Definition of a Lie group from a Lie algebra

$$\{x \in \mathfrak{gl}(n,\mathbb{C})\} \Longrightarrow \left\{X(t) = \mathrm{e}^{xt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n \ \in \mathrm{GL}(n,\mathbb{F})\right\}$$

Jordan decomposition

Jordan decomposition

- $A \in gl(V) \rightsquigarrow A = A_s + A_n$, $[A_s, A_n] = A_s A_n - A_n A_s = 0$
- A_s diagonal matrix (or semisimple)
- A_n nilpotent matrix $(A_n)^m = 0$, the decomposition is unique

$x = x_s + x_n$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^x = e^{(x_s + x_n)} = e^{x_s} e^{x_n}$$

$$x_s \text{ diagonal matrix } = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix} \Rightarrow$$

$$e_s^x \text{ diagonal matrix } = \begin{pmatrix} e^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & 0 & \dots & 0 \\ & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix}$$

$$e^{x_n} = \sum_{k=0}^{m-1} \frac{x_n^k}{k!}$$

Lie algebra of the (Lie) group of isometries

V $\mathbb{F}$-vector space, $b : V \otimes V \longrightarrow \mathbb{C}$ bilinear form

$\mathrm{O}(V, b)$=group of isometries on V i.e.

$$\mathrm{O}(V,b) \ni M : V \underset{\text{lin}}{\longrightarrow} V \rightsquigarrow b(Mu, Mv) = b(u, v)$$

$$\left.\begin{array}{r} M(t) \text{ smooth trajectory } \in \mathrm{O}(V,b) \\ M(0) = I \end{array}\right\} \rightsquigarrow$$

$$M'(0) = T \in \mathfrak{o}(V,\ b) = \text{ Lie algebra}$$

$$\Longrightarrow b(Tu, v) + b(u, Tv) = 0$$

$$\Longrightarrow T^{\text{tr}} b + b\,T = 0$$

group of isometries $\mathrm{O}(V, b) \longrightarrow$ Lie algebra $\mathfrak{o}(V,\ b) = T_{\mathbb{I}}\,(\mathrm{O}(V, b))$

The Lie algebra $\mathfrak{o}(V,\ b)$ is the tangent space of the manifold $\mathrm{O}(V, b)$ at the unity $\mathbb{I}$

Classical Lie Algebras (1)

$\mathtt{A}_\ell$ or $\mathfrak{sl}(\ell+1,\mathbb{C})$ or **s**pecial **l**inear algebra

Def: $(\ell+1)\times(\ell+1)$ complex matrices T with $\mathrm{Tr}\; T = 0$

Dimension $= \dim(\mathtt{A}_\ell) = (\ell+1)^2 - 1 = \ell(\ell+2)$

Standard Basis: E_{ij}, $i \neq j$, $i, j = 1, 2 \ldots, \ell+1$. $H_i = E_{ii} - E_{i+1\, i+1}$

Classical Lie Algebras (2)

C_ℓ or $\mathfrak{sp}(2\ell,\mathbb{C})$ or **symplectic** algebra

Def: $(2\ell)\times(2\ell)$ complex matrices T with $\mathrm{Tr}\,T=0$ leaving infinitesimally invariant the bilinear form

$$b \longleftrightarrow \begin{pmatrix} 0_\ell & \mathbb{I}_\ell \\ -\mathbb{I}_\ell & 0_\ell \end{pmatrix}$$

$$\left.\begin{array}{c} U,\ V \in \mathbb{C}^{2\ell} \\ T \in \mathfrak{sp}(2\ell,\mathbb{C}) \end{array}\right\} \rightsquigarrow b(U,TV)+b(TU,V)=0$$

$$T=\begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{P} & -\mathbb{M}^{\mathrm{tr}} \end{pmatrix} \rightsquigarrow \left\{\begin{array}{c} \mathbb{N}^{\mathrm{tr}}=\mathbb{N}, \\ \mathbb{P}^{\mathrm{tr}}=\mathbb{P} \end{array}\right.$$

$$bT+T^{\mathrm{tr}}b=0 \ \rightsquigarrow \ bTb^{-1}=-T^{\mathrm{tr}}$$

Classical Lie Algebras (3)

B_ℓ or $\mathfrak{o}(2\ell+1,\mathbb{C})$ or (odd) **o**rthogonal algebra

Def: $(2\ell+1)\times(2\ell+1)$ complex matrices T with $\mathrm{Tr}\, T = 0$ leaving infinitesimally invariant the bilinear form

$$b \longleftrightarrow \begin{pmatrix} 1 & \overline{0} & \overline{0} \\ \overline{0}^{\mathrm{tr}} & 0_\ell & \mathbb{I}_\ell \\ \overline{0}^{\mathrm{tr}} & \mathbb{I}_\ell & 0_\ell \end{pmatrix}$$

$\overline{0} = (0,0,\ldots,0)$

$$\left.\begin{array}{c} U,\ V \in \mathbb{C}^{2\ell} \\ T \in \mathfrak{o}(2\ell+1,\mathbb{C}) \end{array}\right\} \rightsquigarrow b(U,TV) + b(TU,V) = 0$$

$$T = \begin{pmatrix} 0 & \overline{b} & \overline{c} \\ -\overline{c}^{\mathrm{tr}} & \mathbb{M} & \mathbb{N} \\ -\overline{b}^{\mathrm{tr}} & \mathbb{P} & -\mathbb{M}^{\mathrm{tr}} \end{pmatrix} \rightsquigarrow \left\{ \begin{array}{c} \mathbb{N}^{\mathrm{tr}} = -\mathbb{N}, \\ \mathbb{P}^{\mathrm{tr}} = -\mathbb{P} \end{array}\right.$$

Classical Lie Algebras (4)

D_ℓ or $\mathfrak{o}(2\ell, \mathbb{C})$ or (even) **o**rthogonal algebra

Def: $(2\ell) \times (2\ell)$ complex matrices T with $\mathrm{Tr}\, T = 0$ leaving infinitesimally invariant the bilinear form

$$b \longleftrightarrow \begin{pmatrix} 0_\ell & \mathbb{I}_\ell \\ \mathbb{I}_\ell & 0_\ell \end{pmatrix}$$

$$\left. \begin{array}{r} U,\ V \in \mathbb{C}^{2\ell} \\ T \in \mathfrak{o}(2\ell, \mathbb{C}) \end{array} \right\} \rightsquigarrow b(U, TV) + b(TU, V) = 0$$

$$T = \begin{pmatrix} \mathbb{M} & \mathbb{N} \\ \mathbb{P} & -\mathbb{M}^{\mathrm{tr}} \end{pmatrix} \rightsquigarrow \left\{ \begin{array}{l} \mathbb{N}^{\mathrm{tr}} = -\mathbb{N}, \\ \mathbb{P}^{\mathrm{tr}} = -\mathbb{P} \end{array} \right.$$

Subalgebras

Definition (Lie subalgebra)

$\mathfrak{h}$ **Lie subalgebra** $\mathfrak{g} \Leftrightarrow$ (i) $\mathfrak{h}$ vector subspace of $\mathfrak{g}$
(ii) $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$

Examples of $\mathfrak{gl}(n, \mathbb{C})$ subalgebras

Diagonal matrices $\mathfrak{d}(n, \mathbb{C})=$ matrices with diagonal elements only

$$[\mathfrak{d}(n, \mathbb{C}), \mathfrak{d}(n, \mathbb{C})] = 0_n \subset \mathfrak{d}(n, \mathbb{C})$$

Upper triangular matrices $\mathfrak{t}(n, \mathbb{C})$,
Strictly Upper triangular matrices $\mathfrak{n}(n, \mathbb{C})$,

$$\mathfrak{n}(n, \mathbb{C}) \subset \mathfrak{t}(n, \mathbb{C})$$

$$[\mathfrak{t}(n, \mathbb{C}), \mathfrak{t}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C}), \quad [\mathfrak{n}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C})$$

$$[\mathfrak{n}(n, \mathbb{C}), \mathfrak{t}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C})$$

ex. $\mathfrak{sl}(2, \mathbb{C})$ Lie subalgebra of $\mathfrak{sl}(3, \mathbb{C})$

$\mathfrak{sl}(2, \mathbb{C})$

Algebra $\mathfrak{sl}(2, \mathbb{C}) = \mathbf{span}\,(h,\ x,\ y)$

$$[h,\ x] = 2x \quad [h,\ y] = -2y \quad [x,\ y] = h$$

---

ex. 2×2 matrices with trace 0

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

---

ex. first derivative operators acting on polynomials of order n

$$h = z\frac{\partial}{\partial z} - \frac{n}{2}, \quad x = z^2\frac{\partial}{\partial z} - nz, \quad y = \frac{\partial}{\partial z}$$

Algebra $\mathfrak{sl}(3,\mathbb{C}) = \mathrm{span}\,(h_1,\ h_2,\ x_1,\ y_1,\ x_2,\ y_2,\ x_3,\ y_3)$

$$
\begin{array}{lll}
[h_1,\ h_2] = 0 & & \\
[h_1,\ x_1] = 2x_1, & [h_1,\ y_1] = -2y_1, & [x_1,\ y_1] = h_1, \\
[h_1,\ x_2] = -x_2, & [h_1,\ y_2] = y_2, & \\
[h_1,\ x_3] = x_3, & [h_1,\ y_3] = -y_3, & \\
[h_2,\ x_2] = 2x_2, & [h_2,\ y_2] = -2y_2, & [x_2,\ y_2] = h_2, \\
[h_2,\ x_1] = -x_1, & [h_2,\ y_1] = y_1, & \\
[h_2,\ x_3] = x_3, & [h_2,\ y_3] = -y_3, & \\
[x_1,\ x_2] = x_3, & [x_1,\ x_3] = 0, & [x_2,\ x_3] = 0 \\
[y_1,\ y_2] = -y_3, & [y_1,\ y_3] = 0, & [y_2,\ y_3] = 0 \\
[x_1,\ y_2] = 0, & [x_1,\ y_3] = -y_2, & \\
[x_2,\ y_1] = 0, & [x_2,\ y_3] = y_1, & \\
[x_3,\ y_1] = -x_2, & [x_3,\ y_2] = x_1, & [x_3,\ y_3] = h_1 + h_2
\end{array}
$$

ex. 3×3 matrices with trace 0

$$
h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
x_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad
x_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad
y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad
y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

Ideals

Definition (Ideal)

Ideal $\mathcal{I}$ is a Lie subalgebra such that $[\mathcal{I},\, \mathfrak{g}] \subset \mathcal{I}$

Center $\mathcal{Z}(\mathfrak{g}) = \{z \in \mathfrak{g} \,:\, [z,\, \mathfrak{g}] = 0\}$

Prop: The center is an ideal.

Prop: The **derived algebra** $\equiv \mathcal{D}\mathfrak{g} = [\mathfrak{g},\, \mathfrak{g}]$ is an ideal.

Prop: If $\mathcal{I}$, $\mathcal{J}$ are ideals $\Rightarrow \mathcal{I} + \mathcal{J}$, $[\mathcal{I},\, \mathcal{J}]$ and $\mathcal{I} \bigcap \mathcal{J}$ are ideals.

Theorem

$\mathcal{I}$ ideal of $\mathfrak{g}$ $\rightsquigarrow$ $\mathfrak{g}/\mathcal{I} \equiv \{\overline{x} = x + \mathcal{I} \,:\, \quad x \in \mathfrak{g}\}$ is a Lie algebra

$$[\overline{x},\, \overline{y}] \equiv \overline{[x,\, y]} = [x,\, y] + \mathcal{I}$$

Simple Ideals

Def: $\mathfrak{g}$ is **simple** $\Leftrightarrow$ $\mathfrak{g}$ has **only** trivial ideals and $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$
(trivial ideals of $\mathfrak{g}$ are the ideals $\{0\}$ and $\mathfrak{g}$)

Def: $\mathfrak{g}$ is **abelian** $\Leftrightarrow [\mathfrak{g}, \mathfrak{g}] = \{0\}$

Prop: $\mathfrak{g}/\left[\mathfrak{g}, \mathfrak{g}\right]$ is abelian

Prop: $\mathfrak{g}$ is **simple Lie algebra** $\Rightarrow \mathcal{Z}(\mathfrak{g}) = 0$ and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Prop: The Classical Lie Algebras $\mathtt{A}_\ell$, $\mathtt{B}_\ell$, $\mathtt{C}_\ell$, $\mathtt{D}_\ell$ are simple Lie Algebras

Direct Sum of Lie Algebras

$\mathfrak{g}_1$, $\mathfrak{g}_1$ Lie algebras

direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g}_1 \times \mathfrak{g}_2$ with the following structure:

$$\alpha(x_1, x_2) + \beta(y_1, y_2) \equiv (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$\rightsquigarrow$ $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ vector space

commutator definition: $[(x_1, x_2)\,,\,(y_1, y_2)] \equiv ([x_1,\, y_1]\,,\,[x_2,\, y_2])$

Prop: $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie algebra

$$\left.\begin{array}{l} (\mathfrak{g}_1, 0) \underset{\text{iso}}{\simeq} \mathfrak{g}_1 \\ (0, \mathfrak{g}_2) \underset{\text{iso}}{\simeq} \mathfrak{g}_2 \end{array}\right\} \rightsquigarrow \left\{\begin{array}{lll} (\mathfrak{g}_1, 0) \bigcap (0, \mathfrak{g}_2) = \{(0,0)\} & \leftrightsquigarrow & \mathfrak{g}_1 \bigcap \mathfrak{g}_2 = 0 \\ [(\mathfrak{g}_1, 0)\,,\,(0, \mathfrak{g}_2)] = \{(0,0)\} & \leftrightsquigarrow & [\mathfrak{g}_1,\, \mathfrak{g}_2] = 0 \end{array}\right.$$

Prop: $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$

Prop: $$\left\{\begin{array}{c} \mathfrak{a},\ \mathfrak{b} \text{ ideals of } \mathfrak{g} \\ \mathfrak{a} \bigcap \mathfrak{b} = \{0\} \\ \mathfrak{a} + \mathfrak{b} = \mathfrak{g} \end{array}\right\} \rightsquigarrow \left\{\mathfrak{a} \oplus \mathfrak{b} \underset{\text{iso}}{\longleftrightarrow} \mathfrak{a} + \mathfrak{b}\right\} \rightsquigarrow \left\{\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{g}\right\}$$

Lie homomorphisms

homomorphism:

$$\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}' \qquad \begin{cases} \text{(i)} & \phi \text{ linear} \\ \text{(ii)} & \phi([x,y]) = [\phi(x), \phi(y)] \end{cases}$$

monomorphism: $\operatorname{Ker}\phi = \{0\}$, **epimorphism**: $\operatorname{Im}\phi = \mathfrak{g}'$

isomorphism: mono+ epi, **automorphism**: iso+ $\{\mathfrak{g} = \mathfrak{g}'\}$

Prop: $\operatorname{Ker}\phi$ is an ideal of $\mathfrak{g}$

Prop: $\operatorname{Im}\phi = \phi(\mathfrak{g})$ is Lie subalgebra of $\mathfrak{g}'$

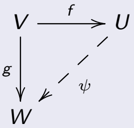
Theorem

$$\left\{ \begin{array}{l} \mathfrak{I} \textit{ ideal of } \mathfrak{g} \\ \textit{canonical map } \ \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{I} \\ \pi(x) = \overline{x} = x + \mathfrak{I} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \textit{canonical map is a} \\ \textit{Lie epimorphism} \end{array} \right\}$$

Linear homomorphism theorem

Theorem

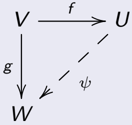
V, U, W linear spaces
$f : V \longrightarrow U$ and $g : V \longrightarrow W$ linear maps

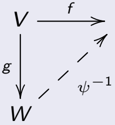


$$\left\{ \begin{array}{c} f \;\; epi \\ Ker\, f \subset Ker\, g \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \exists!\, \psi \; : \\ g = \psi \circ f \end{array} \right\}$$

$$Ker\,\psi = f\,(Ker\, g)$$

Corollary





$$\{ \; f \;\; epi, \quad g \;\; epi, \quad Ker\, f = Ker\, g \; \} \rightsquigarrow \left\{ \; \exists!\, \psi \;\; iso \; : \; g = \psi \circ f, \quad U \underset{\text{iso}}{\simeq} W \; \right\}$$

Lie homomorphism theorem

Theorem

$\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$ *Lie spaces,* $\phi : \mathfrak{g} \underset{\text{epi}}{\longrightarrow} \mathfrak{h}$ *and* $\rho : \mathfrak{g} \longrightarrow \mathfrak{l}$ *Lie homomorphisms*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \\ {\scriptstyle\rho}\downarrow & \swarrow {\scriptstyle\psi} & \\ \mathfrak{l} & & \end{array} \qquad \left\{ \begin{array}{c} \phi \ \text{Lie-epi}, \\ Ker\,\phi \subset Ker\,\rho \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \exists!\,\psi \ : \ \text{Lie-homo} \\ \rho = \psi \circ \phi \end{array} \right\}$$

Corollary

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \\ {\scriptstyle\rho}\downarrow & \swarrow {\scriptstyle\psi} & \\ \mathfrak{l} & & \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \\ {\scriptstyle\rho}\downarrow & \nearrow {\scriptstyle\psi^{-1}} & \\ \mathfrak{l} & & \end{array}$$

$$\left\{ \ \phi \ \text{Lie-epi}, \ \rho \ \text{Lie-epi}, \quad Ker\,\phi = Ker\,\rho \ \right\} \rightsquigarrow \left\{ \ \exists!\,\psi \ : \ \text{Lie-iso}\ \rho = \psi \circ \phi,\ \mathfrak{h} \underset{\text{iso}}{\simeq} \mathfrak{l} \ \right\}$$

First isomorphism theorem

Theorem (First isomorphism theorem)

$\phi \;:\; \mathfrak{g} \longrightarrow \mathfrak{g}'$ *Lie-homomorphism,*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\;\pi\;} & \mathfrak{g}/\mathrm{Ker}\,\phi \\ {\scriptstyle\phi}\big\downarrow & \swarrow{\scriptstyle\phi'} & \\ \phi(\mathfrak{g}) & & \end{array} \quad \Longrightarrow \quad \exists\,!\; \phi' \;:\; \mathfrak{g}/\mathrm{Ker}\,\phi \longrightarrow \phi(\mathfrak{g}) \subset \mathfrak{g}' \quad \textit{Lie-iso}$$

$$\phi(\mathfrak{g}) \underset{\text{iso}}{\simeq} \mathfrak{g}/\mathrm{Ker}\,\phi$$

Noether Theorems

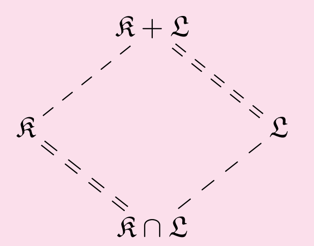
Theorem

$$\left.\begin{array}{c}\mathfrak{K},\ \mathfrak{L}\\ \textit{ideals of } \mathfrak{g}\\ \mathfrak{K}\subset\mathfrak{L}\end{array}\right\} \rightsquigarrow \left\{\begin{array}{ll}\textit{(i)} & \mathfrak{L}/\mathfrak{K}\ \ \textit{ideal of}\ \ \mathfrak{g}/\mathfrak{K}\\ \\ \textit{(ii)} & (\mathfrak{g}/\mathfrak{K})\,/\,(\mathfrak{L}/\mathfrak{K}) \underset{\text{iso}}{\simeq} (\mathfrak{g}/\mathfrak{L})\end{array}\right.$$

Theorem (Noether Theorem (2nd isomorphism theor.))

$$\left\{\begin{array}{c}\mathfrak{K},\ \mathfrak{L}\\ \textit{ideals of } \mathfrak{g}\end{array}\right\} \rightsquigarrow \left\{(\mathfrak{K}+\mathfrak{L})\,/\mathfrak{L} \underset{\text{iso}}{\simeq} \mathfrak{K}/\,(\mathfrak{K}\cap\mathfrak{L})\right\}$$

Parallelogram Low:



Derivations

$\mathcal{A}$ an $\mathbb{F}-$algebra (not necessary associative) with product $\Diamond$

$$\mathcal{A} \times \mathcal{A} \ni (a, b) \longrightarrow a \Diamond b \in \mathcal{A}$$

$\Diamond$ bilinear mapping, ∂ is a derivation of $\mathcal{A}$ if ∂ is a linear map $\mathcal{A} \stackrel{\partial}{\longrightarrow} \mathcal{A}$ satisfying the Leibnitz property:

$$\partial (A \Diamond B) = A \Diamond (\partial(B)) + (\partial(A)) \Diamond B$$

$\mathfrak{Der}(\mathcal{A}) =$ all derivations on $\mathcal{A}$

Prop: $\mathfrak{Der}(\mathcal{A})$ is a Lie Algebra $\subset \mathfrak{gl}(\mathcal{A})$ with commutator

$$[\partial, \partial'] (A) \equiv \partial (\partial'(A)) - \partial' (\partial(A))$$

Derivations on a Lie algebra

$\mathfrak{g}$ a $\mathbb{F}-$ Lie algebra, ∂ is a Lie derivation of $\mathfrak{g}$ if ∂ is a linear map:

$$\mathfrak{g} \ni x \xrightarrow{\partial} \partial(x) \in \mathfrak{g}$$

satisfying the Leibnitz property: $\partial([x, y]) = [\partial(x), y] + [x, \partial(y)]$

$\mathfrak{Der}(\mathfrak{g}) =$ all derivations on $\mathfrak{g}$

Prop: $\mathfrak{Der}(\mathfrak{g})$ is a Lie Algebra $\subset \mathfrak{gl}(\mathfrak{g})$ with commutator $[\partial, \partial'](x) \equiv \partial(\partial'(x)) - \partial'(\partial(x))$

Prop: $\delta \in \mathfrak{Der}(\mathfrak{g}) \rightsquigarrow \delta^n([x, y]) = \sum_{k=0}^{n} \binom{n}{k} \left[\delta^k(x), \delta^{n-k}(y)\right]$

Prop: $\delta \in \mathfrak{Der}(\mathfrak{g}) \rightsquigarrow \mathrm{e}^{\delta}([x, y]) = \left[\mathrm{e}^{\delta}(x), \mathrm{e}^{\delta}(y)\right] \leftrightsquigarrow \mathrm{e}^{\delta} \in \mathrm{Aut}(\mathfrak{g})$

Proposition

$\phi(t)$ smooth monoparametric familly in $\mathrm{Aut}(\mathfrak{g})$ and $\phi(0) = \mathrm{Id} \rightsquigarrow$ $\phi'(0) \in \mathfrak{Der}(\mathfrak{g})$

Adjoint Representation

Definition (Adjoint Representation)

Let $x \in \mathfrak{g}$ and $\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g} \;:\; \mathrm{ad}_x(y) = [x,\, y]$

Prop: $\mathrm{ad}_{[x,\,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$

Prop: $\mathrm{ad}_{\mathfrak{g}} \equiv \bigcup_{x\in\mathfrak{g}} \{\mathrm{ad}_x\}$ Lie-subalgebra of $\mathfrak{Der}\,(\mathfrak{g})$

Definition: **Inner Derivations**$=\mathrm{ad}_{\mathfrak{g}}$

Definition **Outer Derivations**$=\mathfrak{Der}\,(\mathfrak{g})\setminus\mathrm{ad}_{\mathfrak{g}}$

Prop: $\left\{ \begin{array}{c} x \in \mathfrak{g} \\ \delta \in \mathfrak{Der}(\mathfrak{g}) \end{array} \right\} \rightsquigarrow \left\{ [\mathrm{ad}_x,\, \delta] = -\mathrm{ad}_{\delta(x)} \right\} \rightsquigarrow \left\{ \begin{array}{c} \mathrm{ad}_{\mathfrak{g}} \\ \text{ideal of} \\ \mathfrak{Der}(\mathfrak{g}) \end{array} \right\}$

Prop:

$\tau \in \mathrm{Aut}\,(\mathfrak{g}) \leftrightsquigarrow \tau\,([x,\, y]) = [\tau(x),\, \tau(y)] \rightsquigarrow \mathrm{ad}_{\tau(x)} = \tau \circ \mathrm{ad}_x \circ \tau^{-1}$

Matrix Form of the adjoint representation

$$\mathfrak{g} = \mathrm{span}\,(e_1,\, e_2,\, \ldots,\, e_n) = \mathbb{C}e_1 + \mathbb{C}e_2 + \cdots + \mathbb{C}e_n$$

$$[e_i,\, e_j] = \sum_{k=1}^{n} c_{ij}^{k}\, e_k, \qquad c_{ij}^{k} \longleftrightarrow \text{structure constants}$$

Antisymmetry: $c_{ij}^{k} = -c_{ji}^{k}$

Jacobi identity $c_{i\,j}^{m} c_{m\,k}^{\ell} + c_{j\,k}^{m} c_{m\,i}^{\ell} + c_{k\,i}^{m} c_{m\,j}^{\ell} = 0$

$$e^{\ell} = e_{\ell}^{*} \in \mathfrak{g}^{*} \;\rightsquigarrow\; e_{\ell}^{*}.e_p = e^{\ell}.e_p = \delta_p^{\ell}$$

$$\mathbb{P}_i \in \mathfrak{gl}(\mathfrak{g}) \;:\; \boxed{e^{k}.\mathbb{P}_i e_j = (\mathbb{P}_i)_j^{k} = c_{ij}^{k}} \;\rightsquigarrow$$

$$[\mathbb{P}_i,\, \mathbb{P}_j] = \sum_{k=1}^{n} c_{ij}^{k}\, \mathbb{P}_k \;\rightsquigarrow$$

$$\boxed{\mathbb{P}_i = \mathrm{ad}_{e_i}}$$

Representations, Modules

V $\mathbb{F}$-vector space, $u, v, \ldots \in V$, $\alpha, \beta, \ldots \in \mathbb{F}$

Definition

Representation (ρ, V)

$$\mathfrak{g} \ni x \xrightarrow{\rho} \rho(x) \in \mathfrak{gl}(V)$$
$$V \ni v \xrightarrow{\rho(x)} \rho(x)v \in V$$
$$\rho(x) \text{ Lie homomorphism}$$

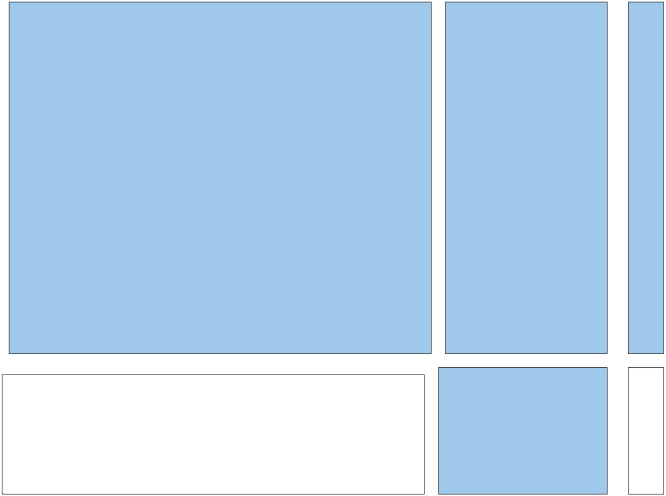
$$\begin{aligned} \rho(\alpha\, x + \beta\, y) &= \alpha\, \rho(x) + \beta\, \rho(y) \\ \rho\left([x,\, y]\right) &= [\rho(x),\, \rho(y)] = \\ &= \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) \end{aligned}$$

$V = \mathbb{C}^n \rightsquigarrow \rho(x) \in M_n(\mathbb{C}) = \mathfrak{gl}(V) = \mathfrak{gl}\,(\mathbb{C}, n)$

$W \subset V$ invariant (stable) subspace

$\leftrightsquigarrow \rho(\mathfrak{g})W \subset W$

$\leftrightsquigarrow$ (ρ, W) is submodule $\leftrightsquigarrow (\rho, W) \prec (\rho, V)$



1.pdf

Theorem (1)

Theorem

W invariant subspace of V $\leftrightsquigarrow (\rho,\ W) \prec (\rho,\ V)$

$(\rho, V) \rightsquigarrow\ \exists!$ *induced representation* $(\overline{\rho}, V/W)$

$$\rho(\mathfrak{g})W \subset W \implies \exists!\,\overline{\rho}(x):\ V/W \longrightarrow V/W$$

$$\implies \begin{array}{ccc} V & \xrightarrow{\pi} & V/W \\ {\scriptstyle\rho(x)}\downarrow & \searrow & \downarrow{\scriptstyle\overline{\rho}(x)} \\ V & \xrightarrow{\pi} & V/W \end{array} \quad \leftrightsquigarrow \quad \overline{\rho}(x)\circ\pi = \pi\circ\rho(x)$$

$\operatorname{Ker}\rho = C_\rho(\mathfrak{g}) = \{x \in \mathfrak{g}\ :\ \rho(x)V = \{0\}\}$ is an ideal

$\operatorname{Ker}\mathrm{ad} = C_{\mathrm{ad}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) =$ center of $\mathfrak{g}$

Reducible and Irreducible representation

(ρ, V) irreducible/simple representation (irrep)

$\leftrightsquigarrow$ $\nexists$ (non trivial) invariant subspaces

$\leftrightsquigarrow$ $\rho(\mathfrak{g})W \subset W \;\Rightarrow\; W = \{0\}$ or V

$\leftrightsquigarrow$ $(\rho, W) \prec (\rho, V) \;\Rightarrow\; W = \{0\}$ or V

(ρ, V) reducible/semisimple representation

$\leftrightsquigarrow$ $V = V_1 \oplus V_2$, $(\rho,\ V_1)$ and $(\rho,\ V_2)$ submodules

$V = V_1 \oplus V_2 \rightsquigarrow\ \rho(\mathfrak{g})V_1 \subset V_1, \quad \rho(\mathfrak{g})V_2 \subset V_2$

---

trivial representation $\leftrightsquigarrow$ $V = \mathbb{F} \leftrightsquigarrow \dim V = 1$

Induced Representation

$(\rho, W) \prec (\rho, V)$

$$\rightsquigarrow \quad \begin{array}{ccc} V & \xrightarrow{\pi} & V/W \\ {\scriptstyle \rho(x)}\downarrow & \searrow & \downarrow{\scriptstyle \overline{\rho}(x)} \\ V & \xrightarrow{\pi} & V/W \end{array} \quad \leftrightsquigarrow \quad \overline{\rho}(x) \circ \pi = \pi \circ \rho(x)$$

$\overline{\rho}(x)$ is the induced representation

Jordan Hölder decomposition (1)

$(\rho, W) \prec (\rho, V)$ and $\boxed{(\overline{\rho}, V/W) \text{ \underline{not} irrep \underline{nor} trivial}}$
$\rightsquigarrow \exists\ (\overline{\rho}, \overline{U}) \prec (\overline{\rho}, V/W) \rightsquigarrow W \subset U = \pi^{-1}(\overline{U}) \subset V$

$$\rightsquigarrow \quad \begin{array}{ccc} V & \xrightarrow{\pi} & V/W \\ {\scriptstyle \rho(x)} \downarrow & \searrow & \downarrow {\scriptstyle \overline{\rho}(x)} \\ V & \xrightarrow{\pi} & V/W \end{array} \quad \leftrightsquigarrow \quad \overline{\rho}(x) \circ \pi = \pi \circ \rho(x)$$

$$(\pi \circ \rho(x))\,(U) = (\overline{\rho}(x) \circ \pi)\,(U) = \overline{\rho}(x)\,\left(\overline{U}\right) \subset \overline{U} = \pi(U) \rightsquigarrow\ \rho(x)(U) \subset U$$

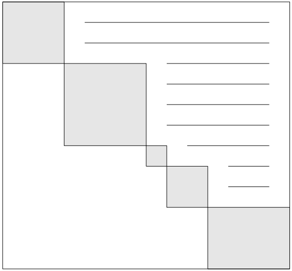
$$\rightsquigarrow \ \boxed{(\rho, W) \prec (\rho, U) \prec (\rho, V)}$$

Theorem (Jordan Hölder decomposition)

(ρ, V) *representation*

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_m = \{0\},\ (\rho, V_i) \succ (\rho, V_{i+1})$$
$$\left(\overline{\rho}_{V_i/V_{i+1}}, V_i/V_{i+1}\right) \text{ irrep or trivial}$$

Jordan Hölder decomposition (2)

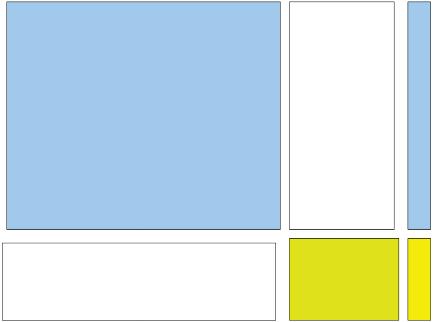


Direct sum of representations

(ρ_1, V_1), (ρ_2, V_2) representations of $\mathfrak{g}$

Def: **Direct sum of representations** $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$

$$(\rho_1 \oplus \rho_2)(x) \ : \ V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$$
$$V_1 \oplus V_2 \ni (v_1, \, v_2) \longrightarrow (\rho_1(x)v_1, \, \rho_2(x)v_2) \in V_1 \oplus V_2$$
$$V_1 \oplus V_2 \ni v_1 + v_2 \longrightarrow \rho_1(x)v_1 + \rho_2(x)v_2 \in V_1 \oplus V_2$$



1b.pdf

Theorem (Jordan- Hölder)

Theorem

Jordan- Hölder $\rightsquigarrow$ *Any representation of a simple Lie algebra is a direct sum of simple representations or trivial representations*

Important

$\Longrightarrow$ **The important is to study the simple representations!**

Theorem (Jordan Hölder decomposition)

(ρ, V) representation

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_m = \{0\},\ (\rho, V_i) \succ (\rho, V_{i+1})$$
$$\left(\overline{\rho}_{V_i/V_{i+1}}, V_i/V_{i+1}\right) \text{ irrep or trivial}$$

$$V = V_{m-1} \oplus V_{m-2}/V_{m-1} \oplus \cdots \oplus V_1/V_2 \oplus V/V_1$$

$$\rho = \rho_{m-1} \oplus \rho_{m-2} \oplus \cdots \oplus \rho_1 \oplus \rho_0$$

$$\rho_{m-1} = \overline{\rho}_{V_{m-1}} \text{ trivial },\ \rho_i = \overline{\rho}_{V_i/V_{i+1}} \text{ simple}$$

Tensor Product of linear spaces- Universal Definition

$\mathbb{F}$ field, $\mathbb{F}$– vector spaces A, B

Definition

Tensor Product $\equiv$ $\boxed{A \otimes_{\mathbb{F}} B}$ is a $\mathbb{F}$-vector space AND a canonical bilinear homomorphism

$$\otimes : A \times B \to A \otimes B,$$

with the universal property.
Every $\mathbb{F}$-bilinear form $\phi : A \times B \to C$,
lifts to a unique homomorphism $\tilde{\phi} : A \otimes B \to C$,
such that $\phi(a, b) = \tilde{\phi}(a \otimes b)$ for all $a \in A,\ b \in B$.

Diagramatically:

$$\begin{array}{ccc} A \times B & \xrightarrow{\ \otimes\ } & A \otimes B \\ & {\scriptstyle \text{bilinear}}\ \searrow^{\phi} & \downarrow {\scriptstyle \exists!\, \tilde{\phi}} \\ & & C \end{array}$$

Tensor product - Constructive Definition

Definition

Tensor product $A \otimes_{\mathbb{F}} B$ can be constructed by taking the free $\mathbb{F}$-vector space generated by all formal symbols

$$a \otimes b, \quad a \in A, \; b \in B,$$

and quotienting by the bilinear relations:

$$\boxed{\begin{gathered}(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b, \\ a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2, \\ r(a \otimes b) = (ra) \otimes b = a \otimes (rb)\end{gathered}}$$

$a_1, a_2 \in A, \; b \in B, \; a \in A, \; b_1, b_2 \in B, \; r \in \mathbb{F}$

Examples: U and V linear spaces

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in U, \quad v = \begin{bmatrix} v_1 \\ u_2 \\ \vdots \\ v_m \end{bmatrix} \in V \quad \rightsquigarrow \quad u \otimes v = \left[\begin{array}{c} u_1 v_1 \\ u_1 v_2 \\ \vdots \\ u_1 v_m \\ \hline u_2 v_1 \\ u_2 v_2 \\ \vdots \\ u_2 v_m \\ \hline \vdots\vdots \\ \hline u_n v_1 \\ u_n v_2 \\ \vdots \\ u_n v_m \end{array}\right] \in U \otimes V$$

Examples (2)

$$\mathbb{U} \in \mathrm{End}(U), \quad \mathbb{V} \in \mathrm{End}(V)$$

$$\mathbb{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & u_{q2} & \cdots & u_{nn} \end{bmatrix}$$

$$\mathbb{V} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mm} \end{bmatrix}$$

$$\mathbb{U} \otimes \mathbb{V} = \left[\begin{array}{c|c|c|c} u_{11}\mathbb{V} & u_{12}\mathbb{V} & \cdots & u_{1n}\mathbb{V} \\ \hline u_{21}\mathbb{V} & u_{22}\mathbb{V} & \cdots & u_{2n}\mathbb{V} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline u_{n1}\mathbb{V} & u_{n2}\mathbb{V} & \cdots & u_{nn}\mathbb{V} \end{array} \right]$$

$$\mathbb{U} \otimes \mathbb{V} \in \mathrm{End}\left(U \otimes V\right)$$

$$\left(\mathbb{U} \otimes \mathbb{V}\right)\left(u \otimes v\right) \equiv \mathbb{U}u \otimes \mathbb{V}v$$

$$\left(\mathbb{U}_1 \otimes \mathbb{V}_1\right)\left(\mathbb{U}_2 \otimes \mathbb{V}_2\right) \equiv \mathbb{U}_1\mathbb{U}_2 \otimes \mathbb{V}_1\mathbb{V}_2$$

Tensor product of representations

$(\rho_1, V_1)\,, \quad (\rho_2, V_2)$ repsesentations of the Lie algebra $\mathfrak{g}$.

Tensor product representation $\left(\rho_1 \overset{\mathrm{L}}{\otimes} \rho_2, V_1 \otimes V_2\right)$

$$\left(\rho_1 \overset{\mathrm{L}}{\otimes} \rho_2\right)(x) \;:\; V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$$

$$v_1 \otimes v_2 \longrightarrow \left(\rho_1 \overset{\mathrm{L}}{\otimes} \rho_2\right)(x)\,(v_1 \otimes v_2)$$

$$\left(\rho_1 \overset{\mathrm{L}}{\otimes} \rho_2\right)(x)\,(v_1 \otimes v_2) = \rho_1(x)v_1 \otimes v_2 + v_1 \otimes \rho_2(x)v_2$$

$$\left(\rho_1 \overset{\mathrm{L}}{\otimes} \rho_2\right)(x) = \rho_1(x) \otimes \mathrm{Id}_2 + \mathrm{Id}_1 \otimes \rho_2(x)$$

Dual representation

V^* dual space of V, $V^* \ni u^* \ : \ V \ni v \longrightarrow u^*.v \in \mathbb{C}$

$(\rho,\ V)$ representation of $\mathfrak{g} \rightsquigarrow \exists!$ **dual representation** $\left(\rho^D,\ V^*\right)$

$$\rho^D(x) \ : \ V^* \ni u^* \longrightarrow \rho^D(x)u^* \in V^*$$

$$\rho^D(x) = -\rho^*(x) \ \rightsquigarrow \ \boxed{\rho^D(x)u^*.v = -u^*.\rho(x)v}$$

$$\rho^D(\alpha x + \beta y) = \alpha\rho^D(x) + \beta\rho^D(y)$$
$$\rho^D\left([x,\ y]\right) = \left[\rho^D(x),\ \rho^D(y)\right]$$

$\mathfrak{sl}(2,\mathbb{C})$ representations (1)

$\mathfrak{sl}(2,\mathbb{C}) = \mathrm{span}\,(h,\ x,\ y)$

$$[h,\ x] = 2x \quad [h,\ y] = -2y \quad [x,\ y] = h$$

(ρ, V) irrep of $\mathfrak{sl}(2,\mathbb{C})$
Jordan decomposition $\rightsquigarrow$ $V = \bigoplus_\lambda V_\lambda$, λ eigenvalue of $\rho(h)$,
$V_\lambda = \mathrm{Ker}\,(\rho(h) - \lambda\mathbb{I})^{n_\lambda}$, $v \in V_\lambda$ $\rightsquigarrow$ $\rho(h)v = \lambda v$, $V_\lambda \bigcap V_\mu = \{0\}$

If v eigenvector of $\rho(h)$ with eigenvalue $\lambda \rightsquigarrow v \in V_\lambda$
$\rightsquigarrow$ $\rho(x)v$ eigenvector with eigenvalue $\lambda + 2 \rightsquigarrow \rho(x)v \in V_{\lambda+2}$
$\rightsquigarrow$ $\rho(y)v$ eigenvector with eigenvalue $\lambda - 2 \rightsquigarrow \rho(y)v \in V_{\lambda-2}$

$\rho(h)v = \lambda v \ \rightsquigarrow \rho(h)\rho(x)v = (\lambda+2)\rho(x)v$ and $\rho(h)\rho(y)v = (\lambda-2)\rho(y)v$

$\exists v_0 \in V$, v eigenvector of $\rho(h)$ such that $\rho(x)v_0 = 0$

If (ρ, V) is a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ then for every $z \in \mathfrak{sl}(2,\mathbb{C}) \rightsquigarrow \mathrm{Tr}\rho(z) = 0$

$\mathfrak{sl}(2,\mathbb{C})$ representations (2)

Let $\rho(h)v_0 = \mu v_0$ and $\rho(x)v_0 = 0 \rightsquigarrow \exists\, n \in \mathbb{N} \,:\, \rho(y)^{n+1}v_0 = 0$

$$\begin{aligned} \rho(x)v_0 &= 0, \quad v_k = \tfrac{1}{k!}\left(\rho(y)\right)^k v_0 \\ \rho(h)v_k &= (\mu - 2k)v_k \\ \rho(y)v_k &= (k+1)v_{k+1} \\ \rho(x)v_k &= (\mu - k + 1)v_{k-1} \end{aligned}$$

$$W = \operatorname{span}(v_0, v_1, \ldots, v_n) \quad \text{and} \quad \rho(z)W \subset W \;\forall\, z \in \mathfrak{sl}(2,\mathbb{C})$$

$$(\rho, W) \;\; \text{submodule of} \;\; (\rho, V)$$

$$\rho(h) = \begin{bmatrix} \mu & 0 & 0 & \ldots & 0 \\ 0 & \mu-2 & 0 & \ldots & 0 \\ 0 & 0 & \mu-4 & \ldots & 0 \\ & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & \mu-2n \end{bmatrix}$$

$$\operatorname{Tr}\rho(h) = 0 \;\rightsquigarrow\; \mu = n$$

Theorem (2.1)

Theorem

(ρ, V) *irrep of* $\mathfrak{sl}(2, \mathbb{C}) \Leftrightarrow V = \mathrm{span}\,(v_0, v_1, \ldots, v_n)$

$$\begin{aligned} \rho(x)v_0 &= 0, \quad v_k = \tfrac{1}{k!}\,(\rho(y))^k\, v_0 \\ \rho(h)v_k &= (n-2k)v_k \\ \rho(y)v_k &= (k+1)v_{k+1} \\ \rho(x)v_k &= (n-k+1)v_{k-1} \end{aligned}$$

$\rho(x)v_k = \mu_k v_k \rightsquigarrow \mu_k = n - 2k$, μ_k are the weights of the representation.

$$\mu_k = n,\ n-2,\ n-4, \ldots, -n+4,\ -n+2,\ -n$$

n is the **highest weight** of the irrep

Theorem (2.2)

$\rho(x)v_0 = 0, \quad v_k = \frac{1}{k!}\left(\rho(y)\right)^k v_0$
$\rho(h)v_k = (n-2k)v_k, \ \rho(y)v_k = (k+1)v_{k+1}, \ \rho(x)v_k = (n-k+1)v_{k-1}$

$$\rho(h) = \begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -n \end{pmatrix}$$

$$\rho(x) = \begin{pmatrix} 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$\rho(y) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \\ 0 & 0 & 0 & \cdots & n & 0 \end{pmatrix}$$

Theorem (2.3)

$$\dim V = n+1$$

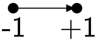
Eigenvalues

$$-n,\ -n+2,\ \ldots,\ n-2,\ n$$

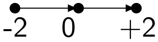
irreducible representation (ρ, V)

$$\rho(x)v_0 = 0, \quad v_k = \frac{1}{k!}\left(\rho(y)\right)^k v_0$$
$$\rho(y)^{n+1}v_0 = 0$$
$$\rho(h)v_k = (n-2k)v_k$$
$$\rho(y)v_k = (k+1)v_{k+1}$$
$$\rho(x)v_k = (n-k+1)v_{k-1}$$

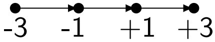
$\alpha = 2$



$\alpha = 2$



$\alpha = 2$



$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

Weyl Theorem

Weyl Theorem

Any representation $V = \bigoplus_n k_n V_n, \; V_n$ irrep

Theorem

For any finite dimensional representation (ρ, V) *the decomposition in direct sum of irreps is unique. The eigenvalues of* $\rho(h)$ *are integers.*

Weyl Theorem $\mathfrak{sl}(2)$

Casimir definition

(ρ, V) a representation of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. The Casimir (operator) K_ρ corresponding to this representation is an element in $\mathrm{End}\, V$ such that

$$K_\rho = \rho^2(h) + 2\left(\rho(x)\rho(y) + \rho(y)\rho(x)\right) \rightsquigarrow [K_\rho\,, \rho(\mathfrak{g})] = \{0\}$$

$V = \bigoplus_\lambda V_\lambda$, λ are eigenvalues of $K_\rho \rightsquigarrow (\rho, V_\lambda)$ is submodule of (ρ, V).

Any representation (module) of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ contains a submodule (ρ, V_n), which is an irreducible representation.

The eigenvalues λ of the Casimir (operator) are given by the formula $\lambda = n(n+2)$, where $n = 0, 1, 2, 3, \dots$.

Let V_λ the submodule corresponding to the eigenvalue $\lambda = n(n+2)$ of the Casimir. If $v \in V_\lambda$ then we can found an irreducible submodule V_n of V_λ and $v \in V_n$. Therefore $V_\lambda = \underbrace{V_n \oplus V_n \oplus \cdots \oplus V_n}_{k_n \text{times}} = k_n V_n$

$\mathfrak{sl}(3,\mathbb{C})$

$\mathfrak{sl}(3,\mathbb{C}) = \mathrm{span}\,(h_1,\ h_2,\ x_1,\ y_1,\ x_2,\ y_2,\ x_3,\ y_3)$

$$
\begin{array}{lll}
[h_1,\ h_2] = 0 & & \\
[h_1,\ x_1] = 2x_1, & [h_1,\ y_1] = -2y_1, & [x_1,\ y_1] = h_1, \\
[h_1,\ x_2] = -x_2, & [h_1,\ y_2] = y_2, & \\
[h_1,\ x_3] = x_3, & [h_1,\ y_3] = -y_3, & \\
[h_2,\ x_2] = 2x_2, & [h_2,\ y_2] = -2y_2, & [x_2,\ y_2] = h_2, \\
[h_2,\ x_1] = -x_1, & [h_2,\ y_1] = y_1, & \\
[h_2,\ x_3] = x_3, & [h_2,\ y_3] = -y_3, & \\
[x_1,\ x_2] = x_3, & [x_1,\ x_3] = 0, & [x_2,\ x_3] = 0 \\
[y_1,\ y_2] = -y_3, & [y_1,\ y_3] = 0, & [y_2,\ y_3] = 0 \\
[x_1,\ y_2] = 0, & [x_1,\ y_3] = -y_2, & \\
[x_2,\ y_1] = 0, & [x_2,\ y_3] = y_1, & \\
[x_3,\ y_1] = -x_2, & [x_3,\ y_2] = x_1, & [x_3,\ y_3] = h_1 + h_2
\end{array}
$$

$\mathfrak{sl}(2,\mathbb{C})$ subalgebras: $\mathrm{span}(h_1,\ x_,1,\ y_1)$, $\mathrm{span}(h_2,\ x_,2,\ y_2)$, $\mathrm{span}(h_1 + h2,\ x_,3,\ y_3)$

Weights-Roots (1)

(ρ, V) is a representation of $\mathfrak{sl}(3,\mathbb{C})$

Def: $\mu = (m_1, m_2)$ is a **weight** $\leftrightsquigarrow$
$\exists\, v \in V \;:\; \rho(h_1)v = m_1 v, \rho(h_2)v = m_2 v$, v is a **weightvector**

Prop: Every representation of $\mathfrak{sl}(3,\mathbb{C})$ has at leat one weight

Prop: $\mu = (m_1, m_2) \in \mathbb{Z}^2$

Weights-Roots (2)

Def: If $(\rho, V) = (\mathrm{ad}, \mathfrak{g})$ weight α is called **root**

$$\alpha = (a_1,\ a_2) \rightsquigarrow \mathrm{ad}_{h_1} z = a_1 z,\ \mathrm{ad}_{h_2} z = a_2 z$$
$$z = x_1,\ x_2,\ x_3,\ y_1,\ y_2,\ y_3$$

	root	rootvector
	α	z_α
α_1	$(2,-1)$	x_1
α_2	$(-1,2)$	x_2
$\alpha_1+\alpha_2$	$(1,1)$	x_3
$-\alpha_1$	$(-2,1)$	y_1
$-\alpha_2$	$(1,-2)$	y_2
$-\alpha_1-\alpha_2$	$(-1,-1)$	y_3

Def: α_1 and α_2 are the **positive simple roots**

Highest weight

Def: $\mathfrak{h} = \mathrm{span}(h_1, h_2)$ is the maximal abelian subalgebra of $\mathfrak{sl}(3, \mathbb{C})$, the **Cartan** subalgebra.

Theorem

The weights $\mu = (m_1, m_2)$ and the roots $\alpha = (a_1, a_2)$ are elements of the $\mathfrak{h}^*$

$$\mu(h_i) = m_i, \quad \alpha(h_i) = a_i$$

$$\rho(h)v = \mu(h)v \rightsquigarrow \rho(h)\rho(z_\alpha)v = (\mu(h) + \alpha(h))\,\rho(z_\alpha)v$$

Def: $\mu_1 \succeq \mu_2 \leftrightsquigarrow \mu_1 - \mu_2 = a\alpha_1 + b\alpha_2$, $a \geq 0$ and $b \geq 0$

Def: μ_0 is **highest weight** if for all weights $\mu \rightsquigarrow \mu_0 \succ \mu$

Highest Weight Cyclic Representation

Definition

(ρ, V) is a **highest weight cyclic representation** with highest weight μ_0 iff

1. $\exists\ v \in V \rightsquigarrow \rho(h)v = \mu_0(h)v$, v is a **cyclic vector**
2. $\rho(x_1)v = \rho(x_2)v = \rho(x_3)v = 0$
3. if (ρ, W) is submodule of (ρ, V) and $v \in W \rightsquigarrow\ W = V$

Theorem (3)

Theorem

If $z_1, z_2, \dots z_n$ are elements of $\mathfrak{sl}(3,\mathbb{C})$ then

$$\rho(z_n)\cdot\rho(z_{n-1})\cdots\rho(z_2)\cdot\rho(z_1) =$$

$$\sum_{p=1}^{n}\Big(\sum_{k_1+k_2+\cdots+k_8=p} c_{k_1,k_2,\dots,k_8}\cdot\rho^{k_3}(y_3)\rho^{k_2}(y_2)\rho^{k_1}(y_1)\rho^{k_4}(h_1)\rho^{k_5}(h_2)\rho^{k_6}(x_3)\rho^{k_7}(x_2)\rho^{k_8}(x_1)\Big) =$$

$$\underbrace{\sum_{k_1+k_2+\cdots+k_8=n} c_{k_1,k_2,\dots,k_8}\cdot\rho^{k_3}(y_3)\rho^{k_2}(y_2)\rho^{k_1}(y_1)\rho^{k_4}(h_1)\rho^{k_5}(h_2)\rho^{k_6}(x_3)\rho^{k_7}(x_2)\rho^{k_8}(x_1)}_{\textit{order } n \textit{ terms}} + \Big(\textit{order } \leq n-1 \textit{ terms}\Big)$$

Corollary (1)

Corollary

If z_1, z_2, $\ldots z_n$ *are elements of* $\mathfrak{sl}(3,\mathbb{C})$ *and* (ρ, V) *is a highest weight cyclic representation with highest weight* μ_0 *and cyclic vector* v *then*

$$\rho(z_n)\cdot\rho(z_{n-1})\cdots\rho(z_2)\cdot\rho(z_1)v=\sum_{p=1}^{n}\Big(\sum_{\ell_1+\ell_2+\ell_3=p} d_{\ell_1,\ell_2,\ell_3}\cdot\rho^{\ell_3}(y_3)\rho^{\ell_2}(y_2)\rho^{\ell_1}(y_1)\Big)v$$

and

$$V=\operatorname{span}\left(\rho^{\ell_3}(y_3)\rho^{\ell_2}(y_2)\rho^{\ell_1}(y_1)v,\quad \ell_i\in\mathbb{N}_0\right)$$

Theorem (4)

Theorem

Every irrep (ρ, V) of $\mathfrak{sl}(3, \mathbb{C})$ is a highest weight cyclic representation with highest weight $\mu_0 = (m_1, m_2)$ and $m_1 \geq 0$, $m_2 \geq 0$

Theorem

The irrep (ρ, V) with highest weight μ_0 is a direct sum of linear subspaces V_μ where

$$\mu = \mu_0 - (k\alpha_1 + \ell\alpha_2 + m\alpha_3) = \mu - (k'\alpha_1 + \ell'\alpha_2)$$

and k, ℓ, m, k' and ℓ' are positive integers.

Corollary (2)

Corollary

The set of all possible weights μ *is a subset of* $\mathbb{Z}^2 \subset \mathbb{R}^2 \rightsquigarrow$
The set of all possible weights corresponds on a discrete lattice in the real plane

Representation $(1,0)$

$$\rho(h_1)=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\quad \rho(h_2)=\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

$$\rho(x_1)=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\quad \rho(x_2)=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\quad \rho(x_3)=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$\rho(y_1)=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\quad \rho(y_2)=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\quad \rho(y_3)=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}$$

The dual of the representation $(1,0)$ is the representation $(0,1)$, i.e $(1,0)^D=(0,1)$ The representation $(1,0)\overset{L}{\otimes}(0,1)$ is the direct sum $(0,0)\oplus(1,1)$

Weight Lattice for $\mathfrak{sl}(3,\mathbb{C})$

Let (ρ, V) the fundamental representation $(1,0)$ then

$$\rho(\mathfrak{sl}(3,\mathbb{C})) \underset{iso}{\cong} \text{ traceless } 3\times 3 \text{ matrices}$$

Def: $(r,s) \overset{\text{def}}{\equiv} \mathrm{Tr}\,(\rho(r)\cdot\rho(s))$

(x,y) is a non-degenerate bilinear form on $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$
$\Leftrightarrow \Big\{(x,\mathfrak{g}) = \{0\} \rightsquigarrow x = 0\Big\}$
$\Leftrightarrow \mathrm{Det}\,(x_i, x_j) \neq 0$, where $x_k = h, x, y$

Proposition

$(.,.)$ is a non-degenerate bilinear form on $\mathfrak{h} \Rightarrow$

$$\Big\{\forall\,\nu\in\mathfrak{h}^* \;\rightsquigarrow\; \exists h_\nu\in\mathfrak{h} \;:\; \nu(h) = (h_\nu, h)\Big\} \;\Rightarrow\; (h)^* \underset{\text{iso}}{\simeq} \in \mathfrak{h}$$

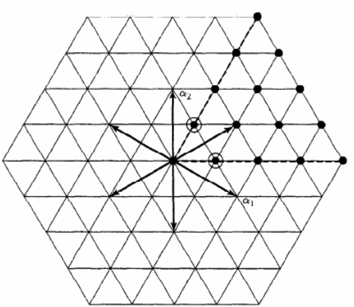
Definition

$\forall\,\mu,\nu\in\mathfrak{h}^* \;\rightsquigarrow\; (\mu,\,\nu) \underset{\text{def}}{\equiv} (h_\mu, h_\nu)$

$(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2)$

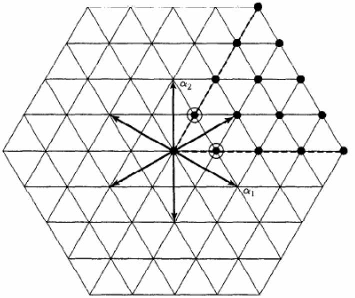
Roots

$$(h_{\alpha_1}, h_1) = \alpha_1(h_1) = 2, \quad (h_{\alpha_1}, h_2) = \alpha_1(h_2) = -1$$
$$(h_{\alpha_2}, h_1) = \alpha_2(h_1) = -1, \quad (h_{\alpha_2}, h_2) = \alpha_2(h_2) = 2$$
$$h_{\alpha_1} = h_1, \quad h_{\alpha_2} = h_2$$
$$(\alpha_1, \alpha_1) = 2, \quad (\alpha_2, \alpha_2) = 2, \quad (\alpha_1, \alpha_2) = -1$$
$$||\alpha_1|| = ||\alpha_2|| = \sqrt{2}, \quad \text{angle } \widehat{\alpha_1\alpha_2} = 120^o$$

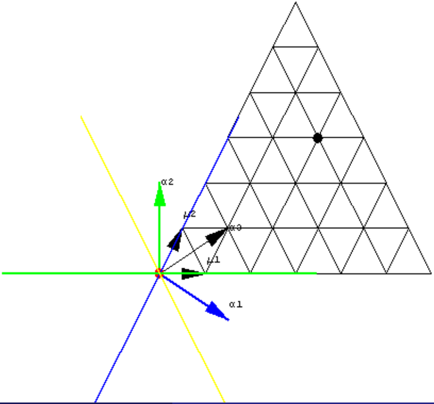


Fundamental weights (1)

$$\mu_1(h_1) = 1, \quad \mu_1(h_2) = 0, \quad \mu_2(h_1) = 0, \quad \mu_2(h_2) = 1$$
$$\mu_1 = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2, \quad \mu_2 = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2$$
$$||\mu_1|| = ||\mu_2|| = \sqrt{\tfrac{2}{3}}, \quad \text{angle } \widehat{\mu_1\mu_2} = 60^o$$
$$\mu = m_1\mu_1 + m_2\mu_2, \quad m_i \in \mathbb{N}_0$$



Fundamental weights (2)



μ weight $\rightsquigarrow \exists\, v \in V \ : \ \rho(h)v = \mu(h)v$
$\rightsquigarrow \exists\, p \in \mathbb{N}_o \ : \ v_0 = \rho^p(x_i)v \neq 0$ and $\rho(x_i)v_0 = 0$,
$\rho(h)v_0 = (\mu(h) + p\alpha_i(h))v_0$
$\rightsquigarrow \exists\, q \in \mathbb{N}_o \ : \ v_1 = \rho(y_i)^{p+q}v_0 \neq$ and $\rho(y_i)v_1 = \rho(y_i)^{p+q+1}v_0 = 0$,
$\rho(h)v_1 = (m(h) - q\alpha_i(h))v_1$.

$$W = \mathrm{span}\left(\rho^{p+q}(y_i)v_0,\ \rho^{p+q-1}(y_i)v_0,\ \ldots,\ \rho(y_i)^2v_0,\ \ldots\ ,\rho(y_i)v_0,\ v_0\right)$$

(ρ, W) is a $\mathfrak{sl}(2,\mathbb{C})$ submodule

Def: If μ is a weight and $\alpha_i,\ i = 1,2,3$ a root, a **chain** is the set of permitted values of the weights

$$\mu - q\alpha_i,\ \mu - (q-1)\alpha_i,\ \ldots, \mu + (p-1)\alpha_i,\ \mu + p\alpha_i$$

The weights of a chain (as vectors) are lying on a line perpendicular to the vertical of the vector α_i. This vertical is a symmetry axis of this chain.

Prop: The weights of a representation is the union of all chains

Theorem (Weyl transform)

Theorem (Weyl transform)

If μ is a weight and α any root then there is another weight given by the transformation

$$S_\alpha(\mu) = \mu - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}\alpha \quad \textit{and} \quad 2\frac{(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$

$\mathfrak{sl}(3, \mathbb{C})$-Representation (4,0)

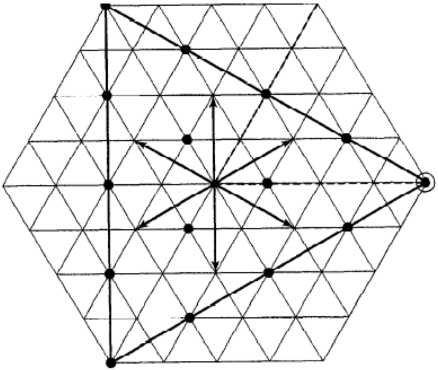


Fig. 5.6. Highest weight (4,0)

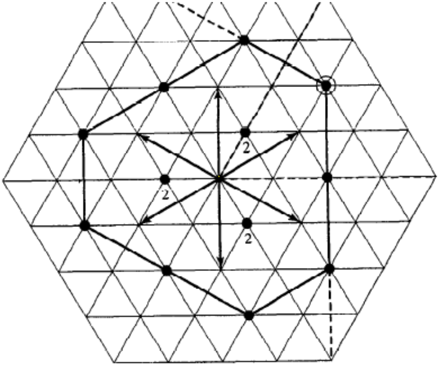


Fig. 5.4. Highest weight (1,2)

$\mathfrak{sl}(3,\mathbb{C})$-Representation (2,2)

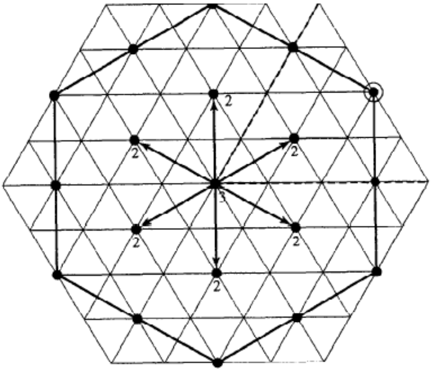


Fig. 5.5. Highest weight (2,2)

Killing form

$(\rho,\ V)$ representation of $\mathfrak{g}$ $\boxed{\mathrm{B}_\rho(x,y) \overset{\mathrm{def}}{\equiv} \mathrm{Tr}\ (\rho(x)\circ\rho(y))}$

Prop: $\boxed{\mathrm{B}_\rho\left([x,\ y]\,,\ z\right) = \mathrm{B}_\rho\left(x, [y,\ z]\right)}$

$\rightsquigarrow\ \mathrm{B}_\rho\left(\mathrm{ad}_y x,\ z\right) + \mathrm{B}_\rho\left(x, \mathrm{ad}_y z\right) = 0$

Definition (Killing form)

$$\mathrm{B}(x,y) \overset{\mathrm{def}}{\equiv} \mathrm{B}_{\mathrm{ad}}(x,y) = \mathrm{Tr}\ (\mathrm{ad}_x \circ \mathrm{ad}_y)$$

$$\mathfrak{g} = \mathrm{span}\left(e_1,\ e_2,\ \ldots,\ e_n\right) \rightsquigarrow$$

$$\mathfrak{g}^* = \mathrm{span}\left(e^1,\ e^2,\ \ldots,\ e^n\right), \quad e^i(e_j) = e^i.e_j = \delta^i_j$$

$$\mathrm{B}(x,y) = \sum_{k=1}^{n} e^k.\mathrm{ad}_x\circ\mathrm{ad}_y e_k = \sum_{k=1}^{n} e^k.\left[x,\ [y,\ e_k]\right]$$

Propositions (1)

Prop: $\delta \in \mathfrak{Der}(\mathfrak{g}) \rightsquigarrow \mathrm{B}(\delta(x), y) + \mathrm{B}(x, \delta(y)) = 0$

Prop: $\tau \in \mathrm{Aut}(\mathfrak{g}) \rightsquigarrow \mathrm{ad}_{\tau x} = \tau \circ \mathrm{ad}_x \circ \tau^{-1} \rightsquigarrow \mathrm{B}(\tau x, \tau y) = \mathrm{B}(x, y)$

Prop: $\mathfrak{k}$ ideal of $\mathfrak{g} \rightsquigarrow x, y \in \mathfrak{k} \Rightarrow \mathrm{B}(x, y) = \mathrm{B}_{\mathfrak{k}}(x, y)$

Prop: $\mathfrak{k}$ ideal of $\mathfrak{g}$, $\quad \mathfrak{k}^{\perp} = \{x \in \mathfrak{g} : \mathrm{B}(x, \mathfrak{k}) = \{0\}\} \rightsquigarrow \mathfrak{k}^{\perp}$ is an ideal.

Derived Series

Def: **Derived Series**

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= \mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \\ \mathfrak{g}^{(2)} &= \mathcal{D}^2(\mathfrak{g}) = \left[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\right] \\ &\ldots\ldots\ldots\ldots\ldots \\ \mathfrak{g}^{(n)} &= \mathcal{D}^n(\mathfrak{g}) = \left[\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}\right] \end{aligned}$$

$\mathfrak{g}^{(k)}$ is an ideal of $\mathfrak{g}$

Definition (Solvable Lie Algebra)

$$\mathfrak{g} \text{ solvable } \rightsquigarrow \exists\, n \in \mathbb{N} \,:\, \mathfrak{g}^{(n)} = \{0\}$$

Exact sequence - extensions

Def: **exact sequence**, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{g}$ Lie spaces

$$\mathfrak{a} \xrightarrow{\mu} \mathfrak{g} \xrightarrow{\lambda} \mathfrak{b} \qquad \operatorname{Im}\mu = \operatorname{Ker}\lambda$$

Def: $\mathfrak{g}$ **extension** of $\mathfrak{b}$ by $\mathfrak{a}$

$$0 \longrightarrow \mathfrak{a} \xrightarrow[1:1]{\mu} \mathfrak{g} \xrightarrow[\text{epi}]{\lambda} \mathfrak{b} \longrightarrow 0$$

Solvable Lie Algebra

Def: **Solvable Lie Algebra** $\mathfrak{g}$ solvable $\leftrightsquigarrow$ $\exists\, n \in \mathbb{N} \;:\; \mathfrak{g}^{(n)} = \{0\}$

Prop: $\mathfrak{g}$ solvable , $\mathfrak{k}$ Lie-subalgebra $\rightsquigarrow$ $\mathfrak{k}$ solvable

Prop: $\mathfrak{g}$ solvable, $\phi \;:\; \mathfrak{g} \stackrel{\text{Lie}-\text{Hom}}{\longrightarrow} \mathfrak{g}'$ $\rightsquigarrow$ $\phi(\mathfrak{g})$ solvable

Prop:
$$\left\{ \begin{array}{c} \mathfrak{I} \text{ solvable ideal} \\ \text{AND} \\ \mathfrak{g}/\mathfrak{I} \text{ solvable} \end{array} \right\} \Rightarrow \;\mathfrak{g} \text{ solvable}$$

Prop: $\mathfrak{a}$ and $\mathfrak{b}$ solvable ideals $\Rightarrow$ $\mathfrak{a} + \mathfrak{b}$ solvable ideal

Theorem (5)

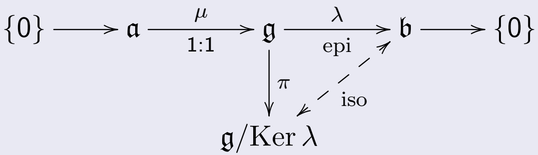
Theorem

$$\mathfrak{a} \xrightarrow{\mu} \mathfrak{g} \xrightarrow{\lambda} \mathfrak{b} \qquad \left\{ \begin{array}{c} \mathfrak{a} \text{ solvable} \\ \text{AND} \\ \mathfrak{b} \text{ solvable} \end{array} \right\} \Rightarrow \mathfrak{g} \text{ solvable}$$

$$\operatorname{Im} \mu = \operatorname{Ker} \lambda$$

Theorem (6)

Theorem



$$\operatorname{Im}\mu = \operatorname{Ker}\lambda$$

$$\mathfrak{g} \text{ solvable } \Rightarrow \left\{ \begin{array}{c} \mathfrak{a} \text{ solvable} \\ \text{AND} \\ \mathfrak{b} \text{ solvable} \end{array} \right\}$$

Equivalent definitions

Theorem

(a) $\mathfrak{g}$ *solvable* $\equiv \mathfrak{g}^{(n)} = \{0\}$

(b) *exists* $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_n = 0$
$\mathfrak{g}_i$ *ideals and* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *abelian.*

(c) *exists* $\mathfrak{g} = \mathfrak{h}'_0 \supset \mathfrak{h}'_1 \supset \mathfrak{h}'_2 \supset \cdots \supset \mathfrak{h}'_p = 0$
$\mathfrak{h}'_i$ *subalgebras of* $\mathfrak{g}$
$\mathfrak{h}'_{i+1}$ *ideal of* $\mathfrak{h}'_i$ *and* $\mathfrak{h}'_i/\mathfrak{h}'_{i+1}$ *abelian.*

(d) *exists* $\mathfrak{g} = \mathfrak{h}''_0 \supset \mathfrak{h}''_1 \supset \mathfrak{h}''_2 \supset \cdots \supset \mathfrak{h}''_q = 0$
$\mathfrak{h}''_i$ *subalgebras of* $\mathfrak{g}$
$\mathfrak{h}''_{i+1}$ *ideal of* $\mathfrak{h}''_i$ *and* $\dim \mathfrak{h}''_i/\mathfrak{h}''_{i+1} = 1$.

$\mathfrak{g}$ solvable Lie algebra $\Rightarrow \exists\, \mathfrak{m}$ ideal such that $\dim(\mathfrak{g}/\mathfrak{m}) = 1$

Theorem

$\mathfrak{g}$ solvable $\rightsquigarrow \mathrm{ad}_{\mathfrak{g}} \subset \mathfrak{t}(n, \mathbb{C})$. There is a basis in $\mathfrak{g}$, where **ALL** *the matrices ad_x, $\forall\, x \in \mathfrak{g}$. are upper triangular matrices.*

$$\mathrm{ad}_x = \begin{pmatrix} \lambda_1(x) & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \lambda_2(x) & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \lambda_3(x) & \bullet & \bullet & \bullet \\ & \dots & \dots & & & \\ & \dots & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & \lambda_n(x) \end{pmatrix}$$

Radical

Definition

$\mathfrak{Rad}(\mathfrak{g})$ = radical = maximal solvable ideal = sum of all solvable ideals

Def: $\mathfrak{Rad}(\mathfrak{g}) = \{0\} \leftrightsquigarrow \mathfrak{g}$ semi-simple

Theorem (Radical Property)

$\mathfrak{Rad}\left(\mathfrak{g}/\mathfrak{Rad}\left(\mathfrak{g}\right)\right) = \left\{\bar{0}\right\} \quad \rightsquigarrow \mathfrak{g}/\mathfrak{Rad}(\mathfrak{g})$ *is semisimple*

Prop: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightsquigarrow \mathfrak{Rad}(\mathfrak{g}) = \mathfrak{Rad}(\mathfrak{g}_1) \oplus \mathfrak{Rad}(\mathfrak{g}_2)$

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^2 = \left[\mathfrak{g}, \mathfrak{g}^1\right], \ldots, \quad \mathfrak{g}^n = \left[\mathfrak{g}, \mathfrak{g}^{n-1}\right] \rightsquigarrow \mathfrak{g}^k \text{ ideals}$$

Definition

$\mathfrak{g}$ nilpotent $\leftrightsquigarrow$ $\mathfrak{g}^n = 0$

$\mathfrak{g}$ nilpotent $\rightsquigarrow$ $\mathfrak{g}$ solvable

Prop: $\mathfrak{g}$ nilpotent $\Leftrightarrow \exists\, m \ : \ \mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} \circ \cdots \circ \mathrm{ad}_{x_m} = 0 \rightsquigarrow (\mathrm{ad}_x)^m = 0$

Prop: $\mathfrak{g}$ nilpotent $\Leftrightarrow \exists\, \mathfrak{g}_i$ ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_r = \{0\}$ $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ and $\dim \mathfrak{g}_i/\mathfrak{g}_{i+1} = 1$

Prop: $\mathfrak{g}$ nilpotent $\Rightarrow$
$\exists$ basis $e_1, e_2, \ldots, e_n$ where ad_x is strictly upper triangular

Prop: $\mathfrak{g}$ nilpotent $\Rightarrow B(x, y) = 0$

Prop: $\mathfrak{g}$ nilpotent $\rightsquigarrow \mathcal{Z}(\mathfrak{g}) \neq \{0\}$

Prop: $\mathfrak{g}/\mathcal{Z}(\mathfrak{g})$ nilpotent $\rightsquigarrow$ $\mathfrak{g}$ nilpotent

Engel's Theorem (1)

Theorem (Engel's theorem)

$\mathfrak{g}$ *nilpotent* $\Leftrightarrow \exists\, n \;:\; (\mathrm{ad}_x)^n = 0$

$\mathfrak{g}$ *nilpotent* $\Leftrightarrow$ $\mathfrak{g}$ *ad-nilpotent*

Lemma

$$\left\{ \begin{array}{c} \mathfrak{g} \subset \mathfrak{gl}(V) \\ \textit{and} \\ x \in \mathfrak{g} \rightsquigarrow x^n = 0 \end{array} \right\} \Longrightarrow \left\{ (\mathrm{ad}_x)^{2n} = 0 \right\}$$

Engel's Theorem in Linear Algebra

Prop: $\left\{ \begin{array}{c} \mathfrak{g} \subset \mathfrak{gl}(V) \\ \text{and} \\ x \in \mathfrak{g} \rightsquigarrow x^n = 0 \end{array} \right\} \Longrightarrow \left\{ \exists\, v \in V \;:\; x \in \mathfrak{g} \rightsquigarrow xv = 0 \right\}$

$\mathfrak{m}$ Lie subalgebra of $\mathfrak{g}$

step #1 $\quad \begin{array}{c} x \in \mathfrak{m} \\ (\mathrm{ad}_x)^m = 0 \end{array} \rightsquigarrow \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{g}/\mathfrak{m} \\ {\scriptstyle \mathrm{ad}x} \downarrow & & \downarrow {\scriptstyle \overline{\mathrm{ad}x}} \\ \mathfrak{g} & \xrightarrow{\pi} & \mathfrak{g}/\mathfrak{m} \end{array} \rightsquigarrow \left(\overline{\mathrm{ad}x}\right)^m = \bar{0}$

step #2 $\quad$ Induction hypothesis $\rightsquigarrow \exists\, \mathfrak{m}$ ideal such that $\dim \mathfrak{g}/\mathfrak{m} = 1$

$\rightsquigarrow \mathfrak{g} = \mathbb{C}x_0 + \mathfrak{m}$

step #3 $\quad$ Induction hypothesis and $\mathfrak{g} = \mathbb{C}x_0 + \mathfrak{m} \rightsquigarrow$

$U = \{v \in V \;:\; x \in \mathfrak{m} \rightsquigarrow xv = 0\} \neq \{0\} \leftrightsquigarrow$

$\exists v \in V \;:\; \forall x \in \mathfrak{g} \rightsquigarrow xv = 0$ and $\mathfrak{g}U \subset U$

Engel's Theorem (2)

Prop : $\left\{ \begin{array}{c} \mathfrak{g} \subset \mathfrak{gl}(V) \text{ and} \\ x \in \mathfrak{g} \rightsquigarrow x^n = 0 \end{array} \right\} \Longrightarrow$
$V = V_0 \supset V_1 \supset V_2 \cdots \supset V_n = 0,\ V_{i+1} = \mathfrak{g} V_i$
$\rightsquigarrow\ (\mathfrak{g})^n\, V = 0 \rightsquigarrow\ x_1 x_2 x_3 \cdots x_n = 0$

$V = V_0 \supset V_1 \supset V_2 \cdots \supset V_n = 0,\ V_{i+1} = \mathfrak{g} V_i$ is a flag

Theorem (Engel's-theorem)

$\mathfrak{g}$ *nilpotent* $\Leftrightarrow \exists\, n\ :\ (\mathrm{ad} x)^n = 0$

Theorem

$\mathfrak{g}$ *nilpotent* $\Leftrightarrow$ *exists a basis in* $\mathfrak{g}$ *such that* *all* *the matrices* $\mathrm{ad}_x,\ x \in \mathfrak{g}$ *are strictly upper diagonal.*

Prop: $\mathfrak{g}$ nilpotent $\Rightarrow\ \{x \in \mathfrak{g} \rightsquigarrow \mathrm{Tr}\, \mathrm{ad}_x = 0\}\ \Rightarrow\ B(x, y) = \mathrm{Tr}\, (\mathrm{ad}_x \mathrm{ad}_y) = 0$

Lie Theorem in Linear Algebra

Theorem (Lie Theorem)

$\mathfrak{g}$ *solvable Lie subalgebra of* $\mathfrak{gl}(V) \Rightarrow \exists\, v \in V \;:\; x \in \mathfrak{g} \rightsquigarrow xv = \lambda(x)v,$
$\lambda(x) \in \mathbb{C}$

---

step $\#1$

Lemma (Dynkin Lemma)

$$\left\{ \begin{array}{c} \mathfrak{g} \subset \mathfrak{gl}(V) \\ \mathfrak{a} \text{ ideal} \\ \exists\, \lambda \in \mathfrak{a}^* \;\; : \\ W = \{v \in V \;:\; \forall\, a \in \mathfrak{a} \rightsquigarrow av = \lambda(a)v\} \end{array} \right\} \Rightarrow \;\; \mathfrak{g}W \in W$$

step # 2 $\boxed{\mathfrak{g} \;\text{ and }\; [\mathfrak{g}, \mathfrak{g}] \neq \{0\}, \quad \Rightarrow \left\{\exists\, \mathfrak{a} \text{ ideal } \;:\; \mathfrak{g} = \mathbb{C}e_0 + \mathfrak{a}\right\}}$

step # 3 $\boxed{\text{induction hypothesis on } \mathfrak{a} \rightsquigarrow \text{Lie theorem on } \mathfrak{g}}$

Lie Theorem

Lemma

$\mathfrak{g}$ *solvable and* (ρ, V) *irreducible module* $\rightsquigarrow \dim V = 1$

Theorem

$\mathfrak{g}$ *solvable and* (ρ, V) *a representation* $\Rightarrow$ *exists a basis in* V *where* *all* *the matrices* $\rho(x)$, $x \in \mathfrak{g}$ *are upper diagonal.*

Theorem

$\mathfrak{g}$ *nilpotent and* (ρ, V) *a representation* $\Rightarrow$ *exists a basis in* V *all* *the matrices* $\rho(x)$, $x \in \mathfrak{g}$ *satisfy the relation* $(\rho(x) - \lambda(x)\mathbb{I})^n = 0$, *where* $\lambda(x) \in \mathbb{C}$ *or the matrices* $(\rho(x) - \lambda(x)\mathbb{I})$ *are strictly upper diagonal.*

Propositions (2)

Prop: $\mathfrak{g}$ solvable $\Rightarrow \mathcal{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent Lie algebra

$$(\rho, W) \prec (\rho, V) \Rightarrow \rho(x) = \left(\begin{array}{c|c} W & * \\ \hline 0 & V/W \end{array}\right)$$

Solvable

$$\rho(x) = \begin{pmatrix} \lambda_1(x) & * & * & * & * & \dots \\ 0 & \lambda_2(x) & * & * & * & \dots \\ 0 & 0 & \lambda_3(x) & * & * & \dots \\ 0 & 0 & 0 & \lambda_4(x) & * & \dots \\ & & \dots & & & \\ & & \dots & & & \end{pmatrix}$$

Nilpotent

$$\rho(x) = \begin{pmatrix} \lambda(x) & * & * & * & * & \dots \\ 0 & \lambda(x) & * & * & * & \dots \\ 0 & 0 & \lambda(x) & * & * & \dots \\ 0 & 0 & 0 & \lambda(x) & * & \dots \\ & & \dots & & & \\ & & \dots & & & \end{pmatrix}$$

Linear Algebra (1)

$A \in \mathrm{End}\, V \simeq M_n(\mathbb{C})$

$p_A(t) = \mathrm{Det}(A - t\mathbb{I})$ = characteristic polynomial

$p_A(t) = (-1)^n \prod_{i=1}^{m} (t - \lambda_i)^{m_i}$, $\sum_{i=1}^{m} m_i = n$, λ_i eigenvalues

$V = \bigoplus_{i=1}^{m} V_i$, $V_i = \mathrm{Ker}\, (A - \lambda_i \mathbb{I})^{m_i} \rightsquigarrow p_A(A) = 0$ and $AV_i \subset V_i$

$Q_i(t) = \dfrac{p_A(t)}{(t - \lambda_i)^{m_i}} \Rightarrow$ If $v \in V_j$ and $j \neq i \rightsquigarrow Q_i(A)v = 0$

Prop: **Chinese remainder theorem**

$Q(t)$, $(Q(0) \neq 0)$ and $P(t)$, $(P(0) \neq 0)$ polynomials with no common divisors, i.e $(Q(t), P(t)) = 1$
$\exists s(t)$, $s(0) = 0$ and $r(t)$ polynomials such that $s(t)Q(t) + r(t)P(t) = 1$

Linear Algebra (2)

Prop: $\exists s_i(t)$ and $r_i(t)$ polynomials such that $s_i(t)Q_i(t) + r_i(t)\,(t-\lambda_i)^{m_i} = 1$

Prop: $S(t) = \sum_i^k \mu_i s_i(t)Q_i(t) \Rightarrow v \in V_j \rightsquigarrow S(A)v = \mu_j v$

Jordan decomposition (1)

Prop: **Jordan decomposition**

$A \in gl(V) \rightsquigarrow A = A_s + A_n$, $[A_s, A_n] = A_sA_n - A_nA_s = 0$
A_s diagonal matrix (or semisimple) and $v \in V_i \leftrightsquigarrow A_s v = \lambda_i v$,
A_n nilpotent matrix $(A_n)^n = 0$, the decomposition is unique

Prop: $\exists\,!$ $p(t)$ and $q(t)$ polynomials such that $A_s = p(A)$ and $A_n = q(A)$

E_{ij} $n \times n$ matrix with zero elements with exception of the ij element, which is equal to 1

$$[E_{ij},\, E_{k\ell}] = \delta_{jk} E_{i\ell} - \delta_{i\ell} E_{kj}$$

Jordan decomposition (2)

Prop: $A = \sum_{i=1}^{n} \lambda_i E_{ii} \rightsquigarrow$ The eigenvalues of ad_A are equal to $\lambda_i - \lambda_j$
$\rightsquigarrow \mathrm{ad}_A E_{ij} = (\lambda_i - \lambda_j)\, E_{ij}$

Prop: $\mathrm{ad}_A = \mathrm{ad}_{A_s} + \mathrm{ad}_{A_n}$ and $(\mathrm{ad}_A)_s = \mathrm{ad}_{A_s}, \quad (\mathrm{ad}_A)_n = \mathrm{ad}_{A_n}$

Cartan Criteria (1)

Lemma

$$\left\{ \begin{array}{c} \mathfrak{g} \subset \mathfrak{gl}(V) \\ x, y \in \mathfrak{g} \rightsquigarrow \mathrm{Tr}(xy) = 0 \end{array} \right\} \Rightarrow [\mathfrak{g}, \mathfrak{g}] = \mathcal{D}\mathfrak{g} = g^{(1)} \text{ nilpotent}$$

Theorem (1st Cartan Criterion)

$\mathfrak{g}$ *solvable* $\Leftrightarrow \mathrm{B}_{\mathcal{D}\mathfrak{g}} = 0$

Corollary

$B(\mathfrak{g}, \mathfrak{g}) = \{0\} \rightsquigarrow \mathfrak{g}$ *solvable*

B is non degenerate, $\mathfrak{a}$ ideal of $\mathfrak{g} \rightsquigarrow \mathfrak{a}^{\perp} = \{x \in \mathfrak{g} \ : \ B(x, \mathfrak{a}) = \{0\}\}$ is an ideal.

Cartan Criteria (2)

Theorem (2nd Cartan Criterion)

$\mathfrak{g}$ *semisimple* $\Leftrightarrow$ *B non degenerate*

Prop

$\mathfrak{a}$ semi-simple ideal of $\mathfrak{g} \rightsquigarrow \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, $\mathfrak{a}^\perp$ is an ideal.

Prop: $\mathfrak{g}$ semisimple, $\mathfrak{a}$ ideal $\Rightarrow \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$

Theorem

$\mathfrak{g}$ *semisimple* $\Leftrightarrow$ $\mathfrak{g}$ *direct sum of simple Lie algebras*

$$\mathfrak{g} = \bigoplus_i \mathfrak{g}_i, \quad \mathfrak{g}_i \textit{ is simple}$$

Cartan Criteria (3)

Theorem

$\mathfrak{g}$ *semisimple* $\Rightarrow \mathfrak{Der}(\mathfrak{g}) = \mathrm{ad}\mathfrak{g}$

Theorem

$\mathfrak{g}$ *semisimple* $\Rightarrow \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathcal{D}\mathfrak{g} = \mathfrak{g}^{(1)}$

$\mathfrak{g}$ semisimple, (ρ, V) a representation $\rightsquigarrow \rho(\mathfrak{g}) \subset \mathfrak{sl}(V)$.

Abstract Jordan decomposition

Prop: $\mathfrak{g}$ semisimple $\rightsquigarrow Z(\mathfrak{g}) = \{0\}$

Prop: $\mathfrak{g}$ semisimple $\rightsquigarrow \mathfrak{g} \underset{\text{iso}}{\simeq} \mathrm{ad}_{\mathfrak{g}}$

Prop: $\partial \in \mathfrak{Der}(\mathfrak{g})$, $\partial = \partial_s + \partial_n$ the Jordan decomposition
$\rightsquigarrow \partial_s \in \mathfrak{Der}(\mathfrak{g})$ and $\partial_n \in \mathfrak{Der}(\mathfrak{g})$

Theorem (Abstract Jordan Decomposition)

$\mathfrak{g}$ *semisimple* $\rightsquigarrow \mathrm{ad}_x = (\mathrm{ad}_x)_s + (\mathrm{ad}_x)_n$ *Jordan decomposition and* $x = x_s + x_n$ *where* $x_s \in \mathfrak{g}$, $(\mathrm{ad}_x)_s = \mathrm{ad}_{x_s}$ *and* $x_n \in \mathfrak{g}$, $(\mathrm{ad}_x)_n = \mathrm{ad}_{x_n}$

Toral subalgebra

$\mathfrak{g}$ is semisimple

Definition semisimple element

x_s is semisimple $\leftrightsquigarrow \exists x \in \mathfrak{g} \ : \ x = x_s + x_n$

x_s is semisimple element $\leftrightsquigarrow$ ad_{x_s} is diagonalizable on $\mathfrak{g}$

x_s and y_s semisimple elements $x_s + y_s$ is semisimple and $[x_s, y_s]$ is semisimple.

Definition Toral/ Cartan subalgebra

Toral subalgebra $\mathfrak{h} = \{x_s, \ x = x_s + x_n \in \mathfrak{g}\}$ i.e the set of all semisimple elements

Theorem

The toral or Cartan subalgebra is abelian

Lie epimorphism

ϕ is a Lie epimorphism $\mathfrak{g} \xrightarrow[\text{epi}]{\phi} \mathfrak{g}'$ and $\mathfrak{g}$ simple $\rightsquigarrow$ $\mathfrak{g}'$ is simple and isomorphic to $\mathfrak{g}$.

ϕ is a Lie epimorphism $\mathfrak{g} \xrightarrow[\text{epi}]{\phi} \mathfrak{g}'$ and $\mathfrak{g}$ semisimple $\rightsquigarrow$ $\mathfrak{g}'$ is semisimple.

Proposition

$\mathfrak{g}$ semisimple, (ρ, V) a representation, $x = x_s + x_n \rightsquigarrow \rho(x) = \rho(x_s) + \rho(x_n)$ and $\rho(x_s) = (\rho(x))_s\,,\ \rho(x_n) = (\rho(x))_n$ the Jordan decomposition of $\rho(x)$ $\rightsquigarrow \rho(x_n)^m = 0$ for some $m \in \mathbb{N}$.

There is some basis in V where all the matrices $\rho(x_s)$ are diagonal and all matrices $\rho(x_n)$ are either strictly upper either lower triangular matrices.

Roots construction

$\mathfrak{g}$ semi-simple algebra,
$\mathfrak{h} = \{x_s,\, x \in \mathfrak{g},\, x = x_s + x_n\}$ Toral subalgebra or Cartan subalgebra,
$\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2 + \cdots \mathbb{C}h_\ell$.
$\mathrm{ad}_\mathfrak{h}$ is a matrix Lie algebra of commuting matrices $\rightsquigarrow$ all the matrices have common eigenvectors
Σ_i= eigenvalues of $h_i \rightsquigarrow \mathfrak{g} = \bigsqcup_{\lambda_i \in \Sigma_i} \mathfrak{g}_{\lambda_i}$, $\mathfrak{g}_{\lambda_i}$ linear vector space
$x \in \mathfrak{g}_{\lambda_i} \rightsquigarrow \mathrm{ad}_{h_i} x = \lambda_i x$ and $\lambda_i \neq \nu_i \ \rightsquigarrow\ \mathfrak{g}_{\lambda_i} \cap \mathfrak{g}_{\nu_i} = \{0\}$

$$x \in \mathfrak{g}_{\lambda_1} \cap \mathfrak{g}_{\lambda_2} \cap \cdots \cap \mathfrak{g}_{\lambda_\ell}, \quad h = c_1 h_1 + c_2 h_2 + \cdots + c_\ell h_\ell$$
$$\mathrm{ad}_h x = (c_1\lambda_1 + c_2\lambda_2 + \cdots + c_\ell\lambda_\ell)\, x$$

$$\mathfrak{h}^* = \mathbb{C}\mu_1 + \mathbb{C}\mu_2 + \cdots + \mathbb{C}\mu_\ell, \quad \mu_i \in \mathfrak{h}^*, \quad \mu_i(h_j) = \delta_{ij}$$

Definition of the roots

$$\mathfrak{h}^* \ni \lambda = \lambda_1\mu_1 + \lambda_2\mu_2 + \cdots + \lambda_\ell\mu_\ell \ \rightsquigarrow\ c_1\lambda_1 + c_2\lambda_2 + \cdots + c_\ell\lambda_\ell = \lambda(h)$$

λ is a root

$$x \in \mathfrak{g}_\lambda = \mathfrak{g}_{\lambda_1} \cap \mathfrak{g}_{\lambda_2} \cap \cdots \cap \mathfrak{g}_{\lambda_\ell} \ \rightsquigarrow\ [h, x] = \mathrm{ad}_h x = \lambda(h) x$$

$$\mathfrak{g} = \bigsqcup_{\lambda} \mathfrak{g}_\lambda, \quad \lambda \neq \mu \rightsquigarrow \mathfrak{g}_\lambda \cap \mathfrak{g}\mu = \{0\}$$

Root space

$\mathfrak{g}$ semisimple algebra, $\mathfrak{h} = \{x_s,\ x \in \mathfrak{g},\ x = x_s + x_n\}$ Toral subalgebra
$\mathrm{ad}_{\mathfrak{h}}$ is a matrix Lie algebra of commuting matrices $\rightsquigarrow$ all the matrices have common eigenvectors

Theorem (Root space)

Exists ***root space*** $\triangle \subset \mathfrak{h}^*$

- $\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup_{\alpha \in \triangle} \mathfrak{g}_\alpha$
- $x \in \mathfrak{g}_\alpha,\ h \in \mathfrak{h} \rightsquigarrow ad_h x = [h,\ x] = \alpha(h)x$
- $\mathfrak{h}$ *is a Lie subalgebra,* $\mathfrak{g}_\alpha$ *are vector spaces*

Prop: $\lambda,\ \mu \in \Delta \rightsquigarrow \left\{ \begin{array}{ll} [\mathfrak{g}_\lambda,\ \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} & \text{if } \lambda + \mu \in \Delta \\ [\mathfrak{g}_\lambda,\ \mathfrak{g}_\mu] = \{0\} & \text{if } \lambda + \mu \notin \Delta \end{array} \right\}$

Semisimple roots

$\mathfrak{g}$ semisimple Lie algebra, $\mathfrak{h}$ Toral/Cartan subalgebra

$B(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0$ if $\lambda + \mu \neq 0$

$h, h' \in \mathfrak{h} \rightsquigarrow B(h, h') = \sum_{\lambda \in \Delta} n_\lambda \lambda(h) \lambda(h'),\ n_\lambda = \dim \mathfrak{g}_\lambda$

$\alpha \in \Delta,\ x \in \mathfrak{g}_\alpha \rightsquigarrow (\mathrm{ad}_x)^m = 0$

Proposition

The Killing form is a non degenerate bilinear form on the Cartan subalgebra $\mathfrak{h}$

$$\{h \in \mathfrak{h},\ B(h, \mathfrak{h}) = 0\} \Rightarrow \{h = 0\}$$

Killing form-non degenerate bilinear form on the Cartan subalgebra

$$\{\forall\, \alpha \in \Delta, \quad \alpha(h) = 0 \rightsquigarrow h = 0\} \;\Leftrightarrow\; \{\mathrm{span}(\Delta) = \mathfrak{h}^*\}$$

$$\{\alpha \in \Delta \Rightarrow -\alpha \in \Delta\} \;\Rightarrow\; \{\alpha \in \Delta \rightsquigarrow \mathfrak{g}_{-\alpha} \neq \{0\}\}$$

Killing form B non degenerate on $\mathfrak{h} \rightsquigarrow \mathfrak{h}^* \ni \phi \xrightarrow[\text{epi}]{1:1} t_\phi \in \mathfrak{h},\ \phi(h) = B(t_\phi, h)$

$x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha} \ \rightsquigarrow [x,\, y] = B(x, y)t_\alpha,\ t_\alpha \in \mathfrak{h}$ is unique

$$[\mathfrak{g}_\alpha,\, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha, \quad \alpha(t_\alpha) = B(t_\alpha, t_\alpha)$$

$$h_\alpha = \frac{2t_\alpha}{B(t_\alpha, t_\alpha)} \ \rightsquigarrow\ S_\alpha = \mathbb{C}h_\alpha + \mathbb{C}x_\alpha + \mathbb{C}y_a \underset{\text{iso}}{\simeq} \mathfrak{sl}(2, \mathbb{C})$$

Propositions (3)

Prop: $\dim \mathfrak{g}_\alpha = 1$

Prop: $\alpha \in \Delta$ and $p\alpha \in \Delta \rightsquigarrow p = -1$

$$S_\alpha = \mathbb{C}h_\alpha + \mathbb{C}x_\alpha + \mathbb{C}y_a \underset{\text{iso}}{\simeq} \mathfrak{sl}(2, \mathbb{C})$$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup_{\alpha \in \Delta_+} (\mathbb{C}x_\alpha + \mathbb{C}y_a) = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

$$\mathfrak{g}_+ = \bigsqcup_{\alpha \in \Delta_+} \mathbb{C}x_\alpha, \quad \mathfrak{g}_- = \bigsqcup_{\alpha \in \Delta_+} \mathbb{C}y_\alpha$$

Example $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$

$$\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2, \quad \mathfrak{g}_+ = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3, \quad \mathfrak{g}_- = \mathbb{C}y_1 + \mathbb{C}y_2 + \mathbb{C}y_3$$

$$\beta \in \Delta, \quad \alpha \in \Delta$$

$$\begin{array}{ll}
\beta \in \Delta \rightsquigarrow \mathfrak{g}_\beta & \\
\beta + \alpha \in \Delta \rightsquigarrow \mathfrak{g}_{\beta+\alpha} & \beta - \alpha \in \Delta \rightsquigarrow \mathfrak{g}_{\beta-\alpha} \\
\beta + 2\alpha \in \Delta \rightsquigarrow \mathfrak{g}_{\beta+2\alpha} & \beta - 2\alpha \in \Delta \rightsquigarrow \mathfrak{g}_{\beta-2\alpha} \\
\dots & \dots \\
\beta + p\alpha \in \Delta \rightsquigarrow \mathfrak{g}_{\beta+p\alpha} & \beta - q\alpha \in \Delta \rightsquigarrow \mathfrak{g}_{\beta-q\alpha} \\
\beta + (p+1)\alpha \notin \Delta & \beta - (q+1)\alpha \notin \Delta
\end{array}$$

String of Roots

$$\{\beta - q\alpha,\ \beta - (q-1)\alpha,\ \dots,\ \beta - \alpha,\ \beta,\ \beta + \alpha,\ \dots,\ \beta + (p-1)\alpha,\ \beta + p\alpha\}$$

$$\mathfrak{g}_\beta^\alpha = \mathfrak{g}_{\beta-q\alpha} \oplus \mathfrak{g}_{\beta-(q-1)\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta-\alpha} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta+\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta+(p-1)\alpha} \oplus \mathfrak{g}_{\beta+p\alpha}$$

Prop: $\left(\mathrm{ad},\ \mathfrak{g}_\beta^\alpha\right)$ is an irreducible representation of $S_a \underset{\mathrm{iso}}{\simeq} \mathfrak{sl}(2,\mathbb{C})$

Prop: $\beta, \alpha \in \Delta \ \rightsquigarrow\ \beta(h_\alpha) = q - p \in \mathbb{Z}, \quad \beta - \beta(h_\alpha)\alpha \in \Delta$

Cartan and Cartan-Weyl basis

Cartan Basis

$$
\boxed{\begin{array}{c}
[E_\alpha, E_{-\alpha}] = t_\alpha, \quad [h_\alpha, E_\alpha] = \alpha(t_\alpha) E_\alpha, \\
[E_\alpha, E_\beta] = k_{\alpha\beta} E_{\alpha+\beta}, \ \text{if } \alpha + \beta \in \Delta \\
B(E_\alpha, E_{-\alpha}) = 1, \quad B(t_\alpha, t_\beta) = \alpha(t_\beta) = \beta(t_\alpha)
\end{array}}
$$

$$
\Delta \ni \alpha \to t_\alpha \ \text{root} \ \leftrightarrow H_\alpha = \frac{2}{\alpha(t_\alpha)} t_\alpha \ \text{coroot}
$$

Cartan Weyl Basis

$$
\boxed{\begin{array}{c}
[X_\alpha, X_{-\alpha}] = H_\alpha, \\
[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, X_{-\alpha}] = -2X_\alpha \\
[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}, \ \text{if } \alpha + \beta \in \Delta \\
\mathbb{C}X_\alpha + \mathbb{C}H_\alpha + \mathbb{C}X_{-\alpha} \underset{\text{iso}}{\simeq} \mathfrak{sl}(2)
\end{array}}
$$

Definitions-Propositions

Def: $(\alpha,\, \beta) \equiv B(t_\alpha,\, t_\beta) = \alpha(t_\beta) = \beta(t_\alpha) = (\beta,\, \alpha)$

Prop: $\ll \beta,\, \alpha \gg \equiv \dfrac{2\,(\beta,\, \alpha)}{(\alpha,\, \alpha)} = q - p \in \mathbb{Z}$

Prop: $w_\alpha(\beta) \equiv \beta - \ll \beta,\, \alpha \gg \alpha, \quad w_\alpha(\beta) \in \Delta$

Def: **Weyl Transform** $w_\alpha : \Delta \longrightarrow \Delta$

Propositions (4)

Prop: $\beta - \alpha \notin \Delta \rightsquigarrow \ll \beta, \alpha \gg < 0,$

Prop: $\beta + \alpha \notin \Delta \rightsquigarrow \ll \beta, \alpha \gg > 0$

Prop: $\beta(h_\alpha) = \ll \beta, \alpha \gg \in \mathbb{Z},\ B(h_\alpha, h_\beta) = \sum_{\gamma \in \Delta} \gamma(h_\alpha)\gamma(h_\beta) \in \mathbb{Z}$

Prop: $B(h_\alpha, h_\alpha) = \sum_{\gamma \in \Delta} (\ll \alpha, \gamma \gg)^2 > 0$

Prop: $\beta \in \Delta$ and $\beta = \sum_{i=1}^{m} c_i \alpha_i,\ \alpha_i \in \Delta \rightsquigarrow c_i \in \mathbb{Q}$

Prop: $\Delta \subset \mathrm{span}_{\mathbb{Q}}(\Delta) = E_{\mathbb{Q}},\ E_{\mathbb{Q}}$ is a $\mathbb{Q}$-euclidean vector space.

Prop: $(\alpha, \beta) \equiv B(t_\alpha, t_\beta) \in \mathbb{Q}\ ,\ (\alpha, \alpha) > 0, \quad (\alpha, \alpha) = 0 \rightsquigarrow \alpha = 0$

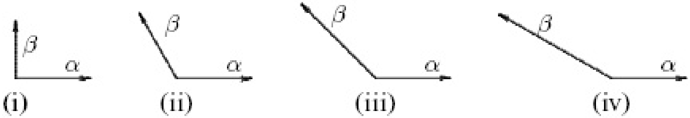
Propositions (5)

Prop: There are not strings with five members $\leftrightsquigarrow$
$\ll \beta, \alpha \gg = 0, \pm 1, \pm 2, \pm 3$

$$(\alpha, \beta) = \|\alpha\| \cdot \|\beta\| \cos\, \theta$$
$$\ll \beta, \alpha \gg = 2\frac{\|\beta\|}{\|\alpha\|} \cos\theta = 0, \pm 1, \pm 2, \pm 3$$
$$\ll \alpha, \beta \gg \cdot \ll \beta, \alpha \gg = 4\cos^2\theta$$

$\ll \alpha, \beta \gg$	$\ll \beta, \alpha \gg$	θ	$\|\beta\|/\|\alpha\|$
0	0	$\pi/2$	
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
-1	-2	$3\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-3	$5\pi/6$	$\sqrt{3}$

Propositions (6)



Simple Roots

Δ=roots is a finite set and $\mathrm{span}_{\mathbb{R}}(\Delta) =<< \Delta >>= E = \mathbb{R}^{\ell}$

Proposition

E is not a finite union of hypersurfaces of dimension $\ell - 1$

$\rightsquigarrow$ E is not the union of hypersurfaces vertical to any root

$$\exists z \in E \ : \ \forall \alpha \in \Delta \rightsquigarrow (\alpha, z) \neq 0$$

Definition of positive/negative roots

$\Delta_{+} = \{\alpha \in \Delta \ : \ (\alpha, z) > 0\}$, positive roots
$\Delta_{-} = \{\alpha \in \Delta \ : \ (\alpha, z) < 0\}$, negative roots

$$\Delta = \Delta_{+} \cup \Delta_{-}, \quad \Delta_{+} \cap \Delta_{-} = \emptyset$$

Definition of Simple roots

$\Sigma = \left\{\beta \in \Delta_{+} \ : \ \text{is not the sum of two elements of } \Delta_{+}\right\}$

$\Sigma \neq \emptyset$

Proposition

$\beta \in \Delta_{+} \ \rightsquigarrow \ \beta = \sum_{\sigma \in \Sigma} n_{\sigma} \sigma, \ n_{\sigma} \geq 0, \ n_{\sigma} \in \mathbb{Z}$

$$\alpha_i \quad \leftrightarrow \quad \epsilon_i = \frac{\alpha_i}{\sqrt{(\alpha_i, \alpha_i)}} = \text{unit vector}$$

$$\Sigma \text{ simple roots} \quad \leftrightarrow \quad \text{Admissible unit roots}$$

Def: Admissible unit vectors $\mathcal{A} = \{\epsilon_1, \epsilon_2, \ldots \epsilon_n\}$

1. ϵ_i linearly independent vectors
2. $i \neq j \rightsquigarrow (\epsilon_i, \epsilon_j) \leq 0$
3. $4\,(\epsilon_i, \epsilon_j)^2 = 0, 1, 2, 3,$

Coxeter diagrams for two points

$\overset{\epsilon_i}{\circ} \qquad\qquad \circ^{\epsilon_j}$	$4\,(\epsilon_i, \epsilon_j)^2 = 0$
$\overset{\epsilon_i}{\circ}$ ——— $\circ^{\epsilon_j}$	$4\,(\epsilon_i, \epsilon_j)^2 = 1$
$\overset{\epsilon_i}{\circ}$ ═══ $\circ^{\epsilon_j}$	$4\,(\epsilon_i, \epsilon_j)^2 = 2$
$\overset{\epsilon_i}{\circ}$ – – – $\circ^{\epsilon_j}$	$4\,(\epsilon_i, \epsilon_j)^2 = 3$

Coxeter diagrams (2)

Prop: $\mathcal{A}' = \mathcal{A} - \{\epsilon_i\}$ admissible

Prop: The number of non zero pairs is less than n

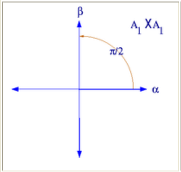
Prop: The Coxeter graph has not cycles

Prop: The maximum number of edges is three

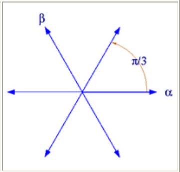
Prop: If $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\} \in \mathcal{A}$ and $2\,(\epsilon_k, \epsilon_{k+1}) = -1$,
$|i - j| > 0 \rightsquigarrow (\epsilon_i, \epsilon_j) = 0$
$\epsilon = \sum_{k=1}^{m} \epsilon_k \Rightarrow \mathcal{A}' = (\mathcal{A} - \{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}) \bigcup \{\epsilon\}$ admissible

Coxeter diagrams (3)

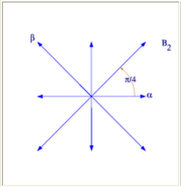
Rank 2 root systems



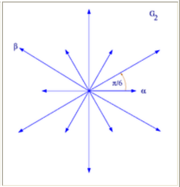
Root system $A_1 \times A_1$



Root system A_2

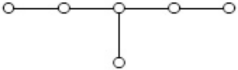
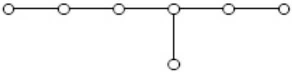
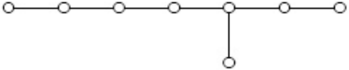


Root system B_2



Root system G_2

Dynkin diagrams

Name	Diagram	Rank
A_l		$l = 1, 2, 3, \ldots$
B_l		$l = 2, 3, 4, \ldots$
C_l		$l = 3, 4, 5, \ldots$
D_l		$l = 4, 5, 6, \ldots$
G_2		$l = 2$
F_4		$l = 4$
E_6		$l = 6$
E_7		$l = 7$
E_8		$l = 8$

Serre relations

$$\mathfrak{g} = \mathfrak{h} \bigoplus \mathfrak{g}_+ \bigoplus \mathfrak{g}_-, \quad \text{simple roots } \Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

$$x_i \in \mathfrak{g}_{\alpha_i},\ y_i \in \mathfrak{g}_{-\alpha_i},\ B(x_i, y_i) = \frac{2}{(\alpha_i, \alpha_i)}$$

$$h_i = h_{\alpha_i} \rightsquigarrow \alpha_i(h_i) = 2,\ \alpha_i(h_j) \in \mathbb{Z}_-$$

Serre Relations

Cartan Matrix: $a_{ij} = \alpha_j(h_i) = 2\frac{(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$

1. $[h_i,\ h_j] = 0$
2. $[h_i,\ x_j] = a_{ij} x_j, \quad [h_i,\ y_j] = -a_{ij} y_j$
3. $[x_i, y_j] = \delta_{ij} h_i$
4. $(\mathrm{ad}_{x_i})^{1-a_{ij}}\, x_j = 0, \quad (\mathrm{ad}_{y_i})^{1-a_{ij}}\, y_j = 0$

Example $\mathfrak{sl}(3, \mathbb{C})$ **Cartan Matrix** $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Weight ordering (1)

Theorem

The weights μ and the roots α are elements of the $\mathfrak{h}^*$

$$h \in \mathfrak{h}, \quad z_\alpha \in \mathfrak{g}_\alpha$$

$$\rho(h)v = \mu(h)v \rightsquigarrow \rho(h)\rho(z_\alpha)v = (\mu(h) + \alpha(h))\,\rho(z_\alpha)v$$

Definition of weight ordering

$$\mu_1 \succeq \mu_2 \leftrightsquigarrow \mu_1 - \mu_2 = \sum_{\alpha \in \Delta_+} k_\alpha \alpha, \ k_\alpha \geq 0$$

Weight ordering (2)

If $(\alpha, \beta) > 0 \ \rightsquigarrow \alpha - \beta \in \Delta$

If $\alpha - \beta \notin \Delta \ \rightsquigarrow \ (\alpha, \beta) < 0$

If $\alpha, \ \beta \ \in \Sigma \ \rightsquigarrow \ (\alpha, \beta) < 0$

Let $\mu_\alpha \in \mathrm{span}\Delta$ such that $\mu_\alpha(h_\beta) = (\mu_\alpha, \beta) = \delta_{\alpha,\beta}$, where $\alpha, \beta \in \Sigma$

$$\mu_o = \sum_{\alpha \in \Sigma} n_\alpha \mu_\alpha, \qquad n_\alpha \in \mathbb{N} \bigcup \{0\}$$

$$\mu = \sum_{\alpha \in \Sigma} m_\alpha \mu_\alpha, \qquad m_\alpha \in \mathbb{Z}_+ \ \text{ or } \ \mathbb{Z}_-, \quad \mu \prec \mu_o$$

Highest weight

Definition of Highest weight

μ_0 is **highest weight** if for all weights *mu* $\rightsquigarrow \mu_0 \succ \mu$

Def: (ρ, V) is a **highest weight cyclic representation** with highest weight μ_0 iff

1. $\exists\, v \in V \rightsquigarrow \rho(h)v = \mu_0(h)v$, v is a **cyclic vector**
2. $\alpha \in \Delta_+ \rightsquigarrow \rho(x_\alpha)v = 0$
3. if (ρ, W) is submodule of (ρ, V) and $v \in W \rightsquigarrow \ W = V$

Highest weight representation

Theorem

Every irrep (ρ, V) *of* $\mathfrak{g}$ *is a highest weight cyclic representation with highest weight* μ_0 *and* $\mu_o(h_\alpha) \in \mathbb{N} \bigcup \{0\}$, $\alpha \in \Delta_+$

Theorem

The irrep (ρ, V) *with highest weight* μ_0 *is a direct sum of linear subspaces* V_μ *where*

$$\mu = \mu_0 - \Big(\sum_{\alpha \in \Delta_+} k_\alpha \alpha \Big)$$

and k_α *are positive integers.*

Chain (2)

μ weight $\rightsquigarrow \exists\, v \in V \;:\; \rho(h)v = \mu(h)v$
$\rightsquigarrow \exists\, p, q \in \mathbb{N}_o \;:\; (\rho, W)$ is a $\mathfrak{g}$ submodule

Def: If μ is a weight and α a root, a **chain** is the set of permitted values of the weights

$$\mu - q\alpha,\; \mu - (q-1)\alpha,\; \ldots, \mu + (p-1)\alpha,\; \mu + p\alpha$$

The weights of a chain (as vectors) are lying on a line perpendicular to the vertical of the vector α. This vertical is a symmetry axis of this chain.

Prop: The weights of a representation is the union of all chains

Theorem (Weyl transform)

If μ is a weight and α any root then there is another weight given by the transformation

$$S_\alpha(\mu) = \mu - 2\frac{(\mu,\alpha)}{(\alpha,\alpha)}\alpha \quad \textit{and} \quad 2\frac{(\mu,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$

Βιβλιογραφία

- J. E. Humphreys *Introduction to Lie Algebras and Representation Theory*,
  Graduate Texts in Mathematics, Springer 1980.
- Alex. Kirillov *An Introduction to Lie Groups and Lie Algebras*,
  Cambridge Univ. Press 2008.
- R. W. CARTER *Lie Algebras of Finite and Affine Type*,
  Cambridge Univ. Press 2005.
- K. Erdmann and M. J. Wildon *Introduction to Lie Algebras*,
  Spinger 2006.
- Hans Samelson, *Notes on Lie Algebras*.
- B. C. Hall *Lie Groups, Lie Algebras and Representations*,
  Grad. Texts in Maths. Springer 2003.
- Andreas Čap, *Lie Algebras and Representation Theory*,
  Univ. Wien, Lecture Notes 2003.
- Alberto Elduque, *Lie algebras*,
  Univ. Zaragoza, Lecture Notes 2005.

Σημείωμα Αναφοράς

Διαθέσιμο από τη δικτυακή διεύθυνση:
http://eclass.auth.gr/courses/OCRS435/

Σημείωμα Αδειοδότησης

Διατήρηση Σημειωμάτων

Οποιαδήποτε αναπαραγωγή ή διασκευή του υλικού θα πρέπει να συμπεριλαμβάνει:

- το Σημείωμα Αναφοράς
- το Σημείωμα Αδειοδότησης
- τη δήλωση Διατήρησης Σημειωμάτων
- το Σημείωμα Χρήσης Έργων Τρίτων (εφόσον υπάρχει)

μαζί με τους συνοδευόμενους υπερσυνδέσμους.

Τέλος Ενότητας

Επεξεργασία:Αναστασία Γρηγοριάδου

Θεσσαλονίκη, Εαρινό Εξάμηνο 2014-2015

Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο
ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ
επένδυση στην κοινωνία της γνώσης
ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ ΚΑΙ ΘΡΗΣΚΕΥΜΑΤΩΝ
ΕΙΔΙΚΗ ΥΠΗΡΕΣΙΑ ΔΙΑΧΕΙΡΙΣΗΣ
ΕΣΠΑ 2007-2013
πρόγραμμα για την ανάπτυξη
ΕΥΡΩΠΑΪΚΟ ΚΟΙΝΩΝΙΚΟ ΤΑΜΕΙΟ
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης