ΑΡΙΣΤΟΤΕΛΕΙΟ
ΠΑΝΕΠΙΣΤΗΜΙΟ
ΘΕΣΣΑΛΟΝΙΚΗΣ

ΑΝΟΙΧΤΑ
ΑΚΑΔΗΜΑΙΚΑ
ΜΑΘΗΜΑΤΑ

Θεωρία Βέλτιστου Ελέγχου

Ενότητα 3: Ακρότατα συναρτήσεων μίας ή πολλών μεταβλητών

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Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο
ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ
επένδυση στην κοινωνία της γνώσης
ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ & ΘΡΗΣΚΕΥΜΑΤΩΝ, ΠΟΛΙΤΙΣΜΟΥ & ΑΘΛΗΤΙΣΜΟΥ
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πρόγραμμα για την ανάπτυξη
ΕΥΡΩΠΑΪΚΟ ΚΟΙΝΩΝΙΚΟ ΤΑΜΕΙΟ
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

Άδειες Χρήσης

Χρηματοδότηση

- Το παρόν εκπαιδευτικό υλικό έχει αναπτυχθεί στα πλαίσια του εκπαιδευτικού έργου του διδάσκοντα.
- Το έργο «Ανοικτά Ακαδημαϊκά Μαθήματα στο Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης» έχει χρηματοδοτήσει μόνο την αναδιαμόρφωση του εκπαιδευτικού υλικού.
- Το έργο υλοποιείται στο πλαίσιο του Επιχειρησιακού Προγράμματος «Εκπαίδευση και Δια Βίου Μάθηση» και συγχρηματοδοτείται από την Ευρωπαϊκή Ένωση (Ευρωπαϊκό Κοινωνικό Ταμείο) και από εθνικούς πόρους.

Περιεχόμενα

- Maximum-Minimum Problems of One Variable Functions.
- Absolute Maximum-Minimum.
- Constrained Problems.
- Maximum-Minimum Problems of Two Variable Functions.
- Absolute Maximum-Minimum.
- Matrix Theory.

Σκοποί Ενότητας

- Βελτιστοποίηση συναρτήσεων μιας μεταβλητής.
- Βελτιστοποίηση συναρτήσεων πολλών μεταβλητών.

Maximum-Minimum Problems of One Variable Functions

- A function $f(x)$ has a **relative maximum** $f(a)$ at the point $(a, f(a))$ if $f(a) \geq f(x)$ for all x in some open interval containing a.
- A function $f(x)$ has a **relative minimum** $f(a)$ at the point $(a, f(a))$ if $f(a) \leq f(x)$ for all x in some open interval containing a.
- **Relative minimum** $\begin{cases} \frac{f(a+h)-f(a)}{h} \geq 0 & if\ h > 0 \\ \frac{f(a+h)-f(a)}{h} \geq 0 & if\ h < 0 \end{cases} \Rightarrow$

$$\begin{cases} f'(a) \geq 0 \\ f'(a) \leq 0 \end{cases} \Rightarrow f'(a) = 0$$

Theorem 1 and Example 1 (1)

Theorem 1. If $f(x)$ is differentiable at a and is defined on an open interval containing a and $f(a)$ is either a relative maximum or a relative minimum of $f(x)$ **THEN** $f'(a) = 0$**.**

Example 1. Find the relative minimum-maximum of

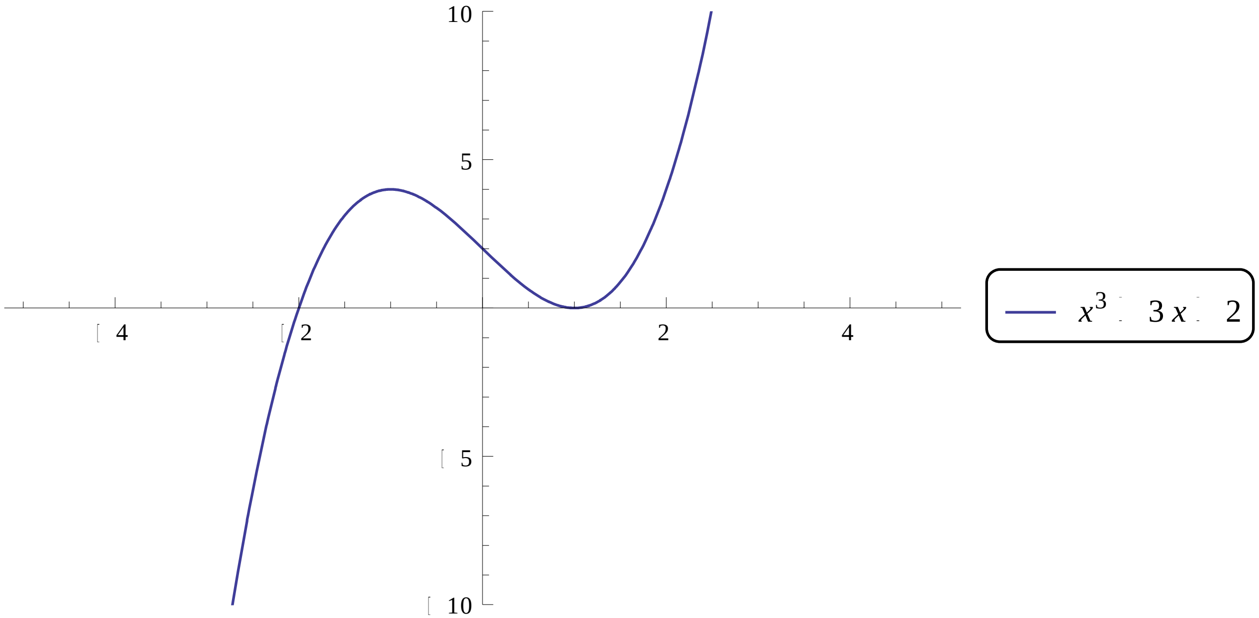
$$f(x) = x^3 - 3x + 2, -\infty \leq x \leq +\infty$$

Since $f(x)$ is differentiable on $\mathbb{R}$ and

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

then we possibly (since Theorem 1 does not give necessary and sufficient conditions) have a relative minimum-maximum at $x = \pm 1$.

Plot of x^3-3x+2



Example 2

Find the relative minimum-maximum of

$$f(x) = x^{2/3}, -2 \leq x \leq +3$$

The derivative of $f(x)$is

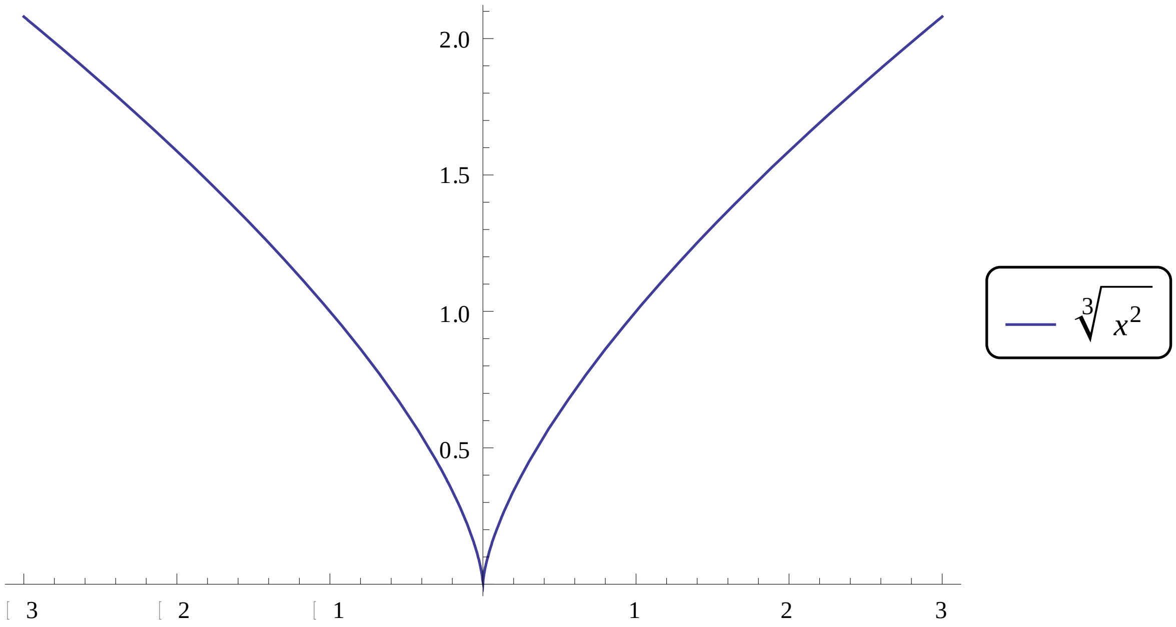
$$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}\frac{1}{x^{\frac{1}{3}}}$$

The derivative does not exist at $x = 0$. Nowhere $f'(x) = 0$. Are there any relative maximum-minimum?

Note that the theorem does not say anything about the points:

- Where there is no derivative.
- The boundaries of our domain.

Plot x^(2/3)



Example 3 (1)

Find the relative minimum-maximum of

$$f(x) = x^3, -3 \leq x \leq +3$$

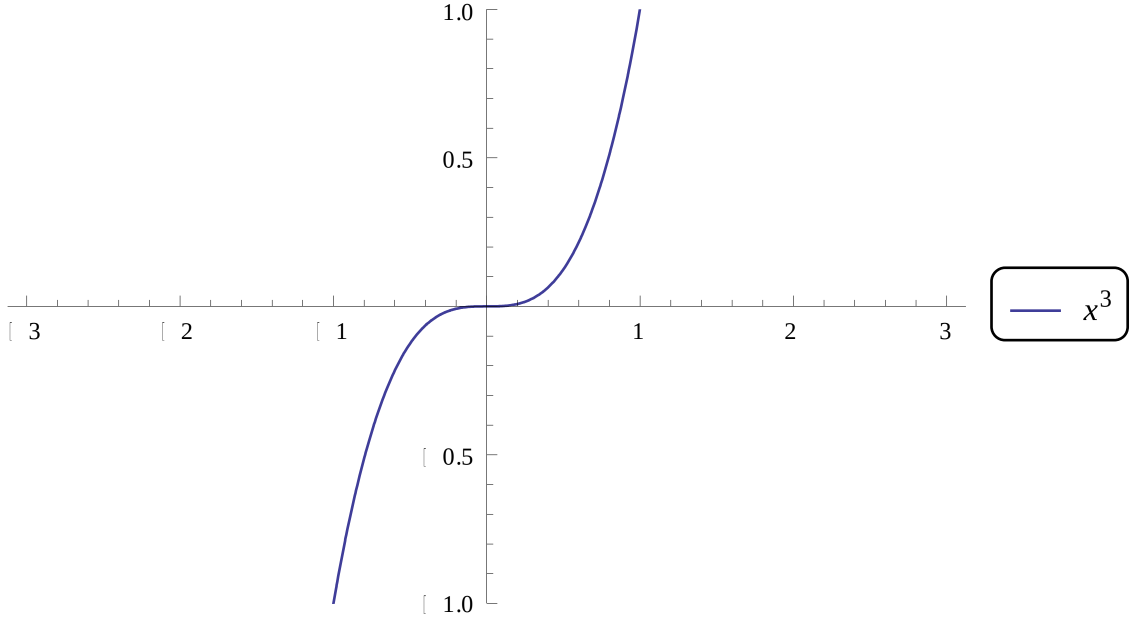
The derivative of $f(x)$is

$$f'(x) = 3x^2$$

Note that $f'(x) = 0 \Rightarrow x = 0$ but we don't have any relative minimum-maximum at $x = 0$.

Note: The converse of the Theorem 1 is not always True!

Example 3 (2)



The first derivative test

Let a be a critical value of $f(x)$ and suppose that $f(x)$ is differentiable for all values of x near to a (but not necessarily at a). For values of x near a,

A. If $f'(x) \geq 0$ for $x < a$ and $f'(x) \leq 0$ for $x > a$, then $f(a)$ is a relative maximum.

B. If $f'(x) \leq 0$ for $x < a$ and $f'(x) \geq 0$ for $x > a$, then $f(a)$ is a relative minimum.

C. If $f'(x)$ has the same sign on both sides of $x = a$, then $f(a)$ is neither minimum nor maximum.

Example 4 (1)

Find the relative minimum-maximum of

$$f(x) = x^3 - 3x + 2, -\infty \leq x \leq +\infty$$

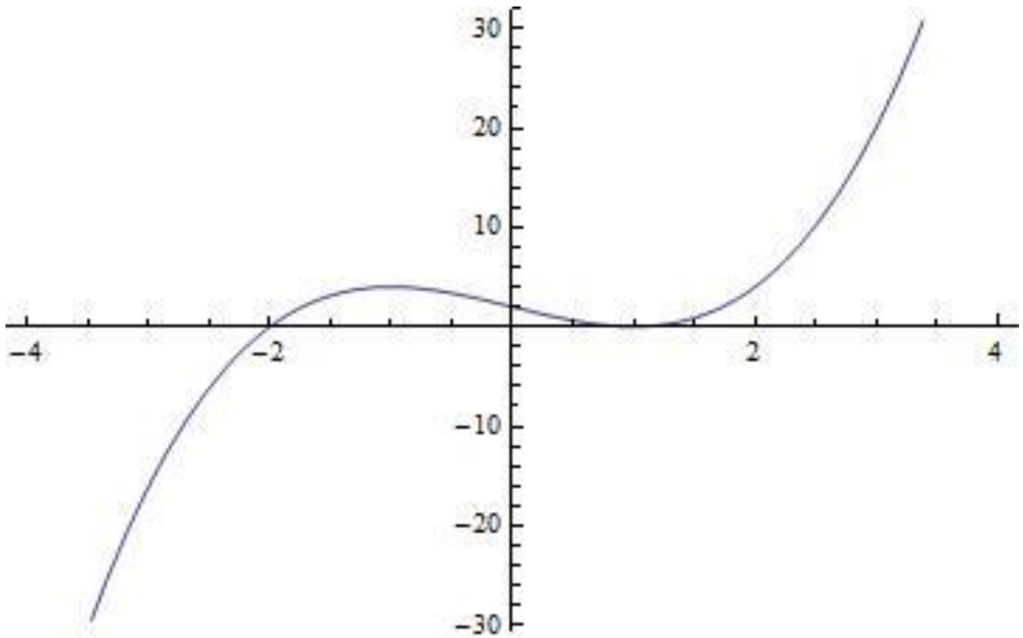
Since $f(x)$ is differentiable on $\mathbb{R}$ and

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

$$\begin{cases} f'(x) \geq 0, & \forall x \in (-\infty, -1] \\ f'(x) \leq 0 & \forall x \in [-1,1] \end{cases} \Rightarrow x = -1 \ (maximum)$$

$$\begin{cases} f'(x) \leq 0, & \forall x \in [-1,1] \\ f'(x) \geq 0 & \forall x \in [1, +\infty) \end{cases} \Rightarrow x = 1 \ (minimum)$$

Example 4 (2)



Example 5

Find the relative minimum-maximum of

$$f(x) = x^3, -3 \leq x \leq +3$$

The derivative of $f(x)$is

$$f'(x) = 3x^2 \geq 0$$

Since $f'(x)$ has the same sign on both sides of $x = 0$, then $f(0)$ is neither minimum or maximum.

The second derivative test

Suppose that $f(x)$ is twice differentiable function in some interval about a and suppose that $f'(a) = 0$.

A. If $f''(a) < 0$ then $f(a)$ is a relative maximum of $f'(x)$.

B. If $f''(a) > 0$ then $f(a)$ is a relative minimum of $f'(x)$.

C. If $f''(a) = 0$ then the test is inconclusive.

   I. If $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0, f^{(n)}(a) \neq 0$ then

      a) a is an extreme point iff n is even (like $f(x) = x^2$). Further when n is even, there is a relative maximum at $x = a$ if $f^{(n)}(a) < 0$ and a relative minimum if $f^{(n)}(a) > 0$.

      b) a is an inflexion if is odd (like $f(x) = x^3$).

Example 6

Find the relative minimum-maximum of

$$f(x) = x^3 - 3x + 2, -\infty \leq x \leq +\infty$$

Since $f(x)$ is differentiable on $\mathbb{R}$ and

$$f'(x) = 3x^2 - 3 \Rightarrow f'(-1) = f'(1) = 0$$

$$f''(x) = 6x$$

$$\Rightarrow \begin{cases} f''(-1) = -6 < 0 & f(-1)\ relative\ maximum \\ f''(1) = 6 > 0 & f(1)\ relative\ minimum \end{cases}$$

Absolute Maximum-Minimum

- The absolute maximum of the function $f(x)$ is a value $f(a)$ such that $f(a) \geq f(x)$ for all values of x in the domain of $f(x)$.
- The absolute minimum of the function $f(x)$ is a value $f(a)$ such that $f(a) \leq f(x)$ for all values of x in the domain of $f(x)$.

How to find the absolute maximum-minimum of the continuous function $f(x)$ on $[a, b]$?

How to find the absolute maximum-minimum of the continuous function $f(x)$ on $[a, b]$?

1. Find all the critical values of $f(x)$ i.e. $f'(x) = 0$.
2. Find the values $f(a), f(b)$.
3. Find the values of $f(x)$ for the points where there is no derivative of $f(x)$.
4. Of the above values, the largest is the absolute maximum, and the smallest is the absolute minimum.

The cake-pan: Optimization

A manufacturing company plans to make microwave-safe cake pans by cutting squares out of the corners of a 12-inch by 12-inch piece of plastic, and then bending the sides up. The seam along each corner will be fused to finish the pan.

The manufacturer wishes to determine what size square should be cut from the corners to make a pan of the greatest possible volume.

Solution (1)

- **Step 1.** Understand the problem
- V=volume of the pan.
- x= the length of a side of the square base.
- h=the length of a side of the corner to be cut out=the height of the pan
- $V = x^2 h$ and $x + 2h = 12$
- **Step 2.** Form a mathematical statement of the problem.

$$x = 12 - 2h$$

$$V(h) = (12 - 2h)^2 h, 0 \leq h \leq 6$$

We seek the maximum value of $V(h)$ for $0 \leq h \leq 6$.

Solution (2)

- **Step 3.** Determine the maximum- minimum

Find the derivative of $V(h)$:

$$V'(h) = 2(12-2h)(-2)h + (12-2h)^2$$
$$= (12-2h)(-4h+12-2h) = (12-2h)(-6h+12)$$

Therefore $V'(h) = 0$ for $h \in \{2,6\}$. We have also that:

$$V''(h) = (-2)(-6h+12) + (12-2h)(-6) =$$
$$= 24h - 96 = 24(h-4)$$

and thus $V''(2) = -48 < 0, V''(6) = 48 > 0$.

Solution (3)

Therefore we have a relative maximum for $x = 2$. We have also that $V(0) = 0, V(6) = 0$. Thus the largest of the values $V(0), V(2), V(6)$ is

$$V(12 - 2 * 2)^2 2 = 8^2 * 2 = 128in^3$$

which is the absolute maximum.

- **Step 4.** Answer the question

Since we have the absolute maximum for we have that and therefore the dimensions of the largest tray are 8 inches by 8 inches, a very common size for microwave trays and cake pans.

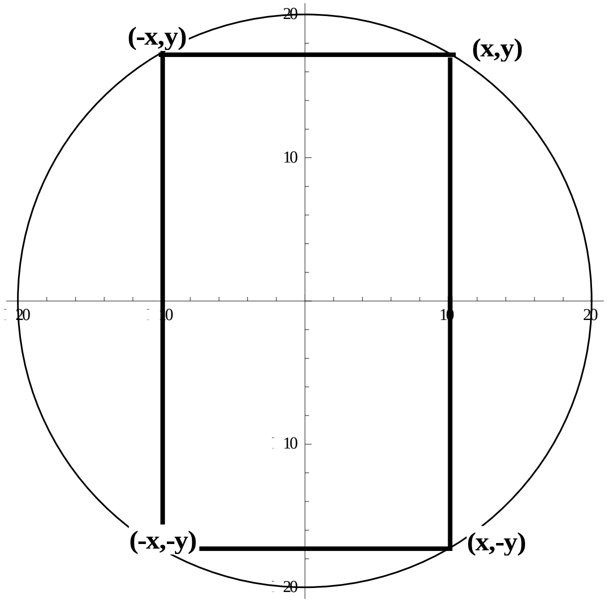
What about a non-square base?

Rectangle in a circle (1)

What is the rectangle of largest area that can be cut from a circle of radius 20 inches?

- **Step 1.** Understand the problem.

Let be one of the corners of the rectangle, as shown below.

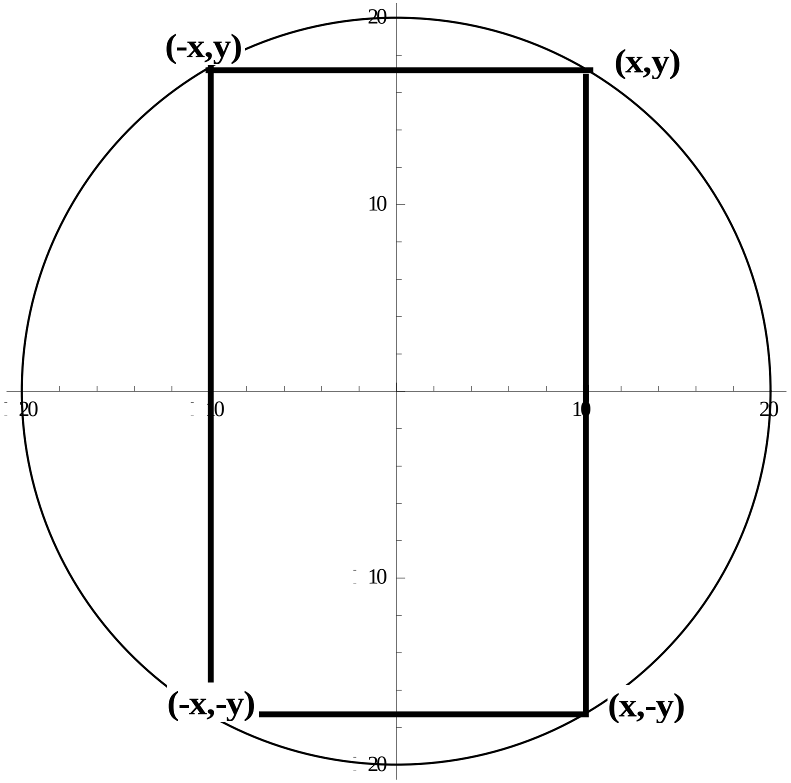


Rectangle in a circle (2)

What is the rectangle of largest area that can be cut from a circle of radius 20 inches?

- Step 1. Understand the problem.

Let be one of the corners of the rectangle, as shown below.



$$x^2 + y^2 = 20^2$$
$$E = (2x) * (2y) = 4xy$$

Rectangle in a circle (3)

- **Step 2.** Form a mathematical statement of the problem.

$$x = \sqrt{20^2 - y^2}, 0 \leq y \leq 20$$

$$E(y) = 4xy = 4y\sqrt{20^2 - y^2}$$

We seek the maximum value of $E(y)$ for $0 \leq y \leq 20$.

- **Step 3.** Determine the maximum-minimum.

Find the derivative of $E(y)$:

$$E'(y) = 4\sqrt{20^2 - y^2} + 4y\frac{1}{2}\frac{1}{\sqrt{20^2 - y^2}}(-2y) =$$

$$= 4\sqrt{20^2 - y^2} - \frac{4y^2}{\sqrt{20^2 - y^2}} = \frac{4(20^2 - y^2) - 4y^2}{\sqrt{20^2 - y^2}} = 4\frac{(20^2 - 2y^2)}{\sqrt{20^2 - y^2}}$$

Rectangle in a circle (4)

Thus $E'(y) = 0 \Rightarrow 20^2 - 2y^2 = 0 \Rightarrow y = \pm 10\sqrt{2}$

We also have that

$$E''(y) = \left(4\frac{(20^2-2y^2)}{\sqrt{20^2-y^2}}\right)' = \frac{8y(y^2-600)}{(400-y^2)^{\frac{3}{2}}}$$

and thus $E''(10\sqrt{2}) = -16 < 0, E''(-10\sqrt{2}) = 16 > 0$

Therefore we have a relative maximum for $y = 10\sqrt{2}$.

We have also that $E(0) = 0, E(20) = 0$.

Thus the largest of the values $E(0), E(10\sqrt{2}), E(20)$ is

$$E(10\sqrt{2}) = 800in^2$$

which is the absolute maximum.

Rectangle in a circle (5)

- **Step 4.** Answer the question.

Since we have the absolute maximum for $\mathrm{y} = 10\sqrt{2}$, we have that $\mathrm{x} = \mathrm{x} = \sqrt{20 - y^2} = 10\sqrt{2}$ and therefore the dimension of the rectangle are $10\sqrt{2}$ inches by $10\sqrt{2}$ inches (square).

Constrained Problems

- Minimize-Maximize $Q(x, y)$ subject to constraints $C(x, y) = K \in \mathbb{R}$.

$$\Updownarrow$$

- Minimize-Maximize $Q\big(x, y(x)\big)$ subject to constraints $C\big(x, y(x)\big) = K \in \mathbb{R}$.

$$\frac{d}{dx} Q\big(x, y(x)\big) = 0, \frac{d}{dx} C\big(x, y(x)\big) = 0$$

Example 7 (1)

Minimize

$$E = (2x) * (2y) = 4xy$$

subject to constraints

$$x^2 + y^2 = 20^2.$$

Solution. We have that $E\big(x, y(x)\big) = 4xy(x)$ and $x^2 + y(x)^2 = 20^2$, and thus

$$\frac{d}{dx}E\big(x, y(x)\big) = 0 \Leftrightarrow 4y(x) + 4x\frac{dy}{dx} = 0$$

$$\frac{d}{dx}\left(x^2 + y(x)^2\right) = \frac{d}{dx}\left(20^2\right) \Rightarrow 2x + 2y(x)\frac{dy}{dx} = 0$$

The cases $x = 0$ and $\mathrm{y} = 0$ do not produce maxima, so we may

Example 7 (2)

assume that $x > 0$ and $y(x) > 0$.

With these assumptions the two equations become

$$\frac{dy}{dx} = -\frac{y(x)}{x} \text{ and } \frac{dy}{dx} = -\frac{x}{y(x)}$$

respectively. Thus

$$\frac{dy}{dx} = -\frac{y(x)}{x} = -\frac{x}{y(x)} \Leftrightarrow x^2 = y^2(x)$$

The only first-quadrant solution of the equations

$$x^2 = y^2(x) \text{ and } x^2 + y(x)^{\,2} = 20^2$$

is the point $\left(10\sqrt{2}, 10\sqrt{2}\right)$ which determines a square with edge length $20\sqrt{2}$. (the same solution with previous example)

Maximum-Minimum Problems of Two Variable Functions

- A function $f(x, y)$ has a **relative maximum** $f(a, b)$ at the point $\big((a, b), f(a, b)\big)$ if $f(a, b) \geq f(x, y)$ for all (x, y) in some rectangular region about (a, b).
- A function $f(x, y)$ has a **relative minimum** $f(a, b)$ at the point $\big((a, b), f(a, b)\big)$ if $f(a, b) \leq f(x, y)$ for all (x, y) in some rectangular region about (a, b).

Theorem 2

Theorem 2. (Necessary Conditions) Suppose that $f(x, y)$ attains a relative maximum or a relative minimum value at the point (a, b) and the partial derivatives

$$f_x(a, b) \text{ and } f_y(a, b)$$

both exist. Then

$$f_x(a, b) = 0 = f_y(a, b)$$

Example 8 (1)

Find the relative minimum-maximum of

$$f(x,y) = x^2 + y^2, -\infty \leq x, y \leq +\infty$$

Since the partial derivatives

$$f_x = \frac{\partial f}{\partial x} = 2x \text{ and } f_y = \frac{\partial f}{\partial y} = 2y$$

both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

Then we possible (since Theorem 2 does not give necessary and sufficient conditions) have a relative a relative minimum-maximum at $(x,y) = (0,0)$.

Since

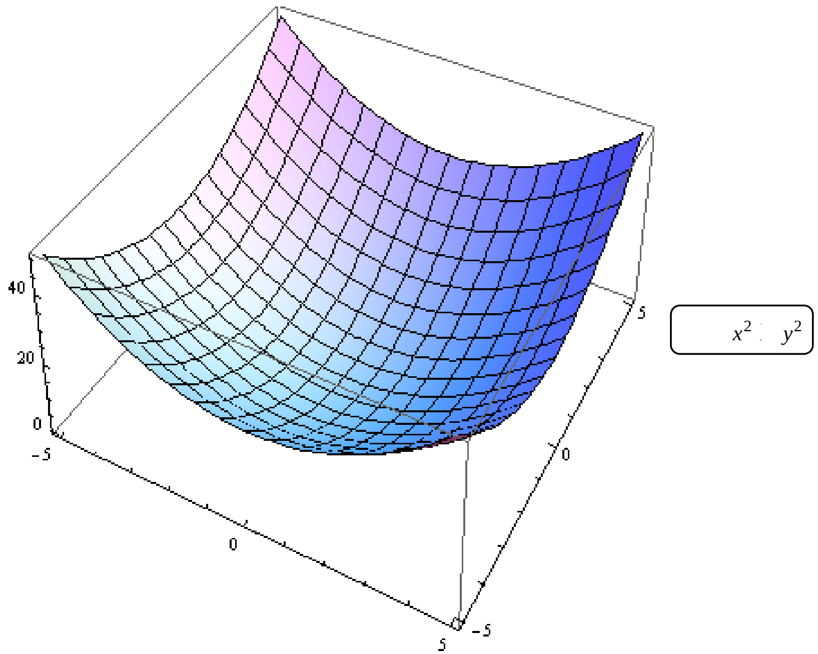
$$f(x,y) - f(0,0) = x^2 + y^2 - 0 = x^2 + y^2 \geq 0$$

Example 8 (2)

we have that

$$f(x,y) \geq f(0,0)$$

Thus we have a minimum at $(x,y) = (0,0)$.



Example 9 (1)

Find the relative minimum-maximum of

$$f(x,y) = x^2 - y^2, -\infty \leq x, y \leq +\infty$$

Since the partial derivatives

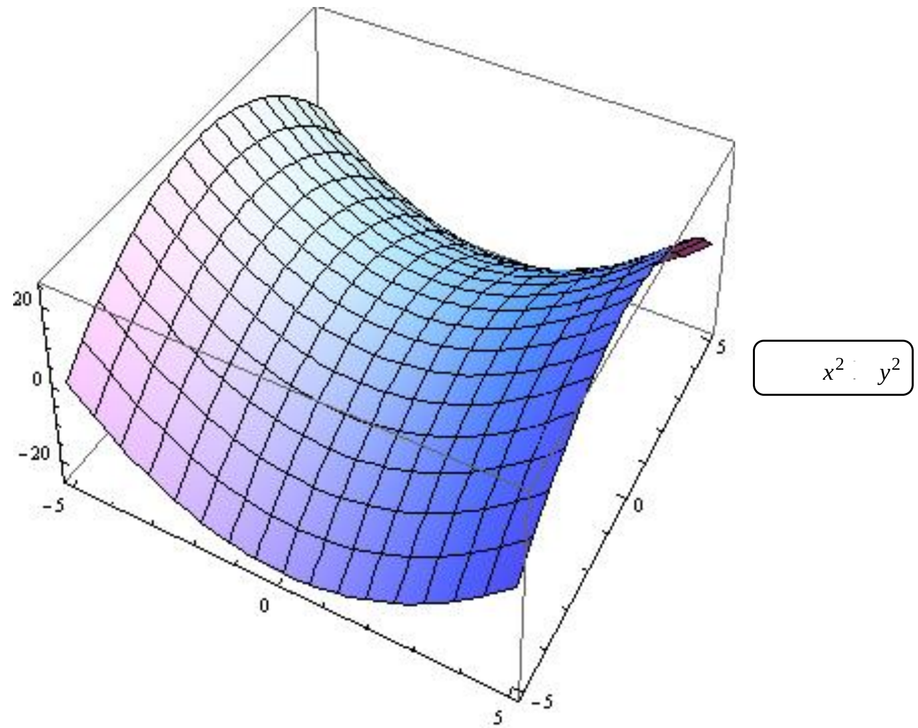
$$f_x = \frac{\partial f}{\partial x} = 2x \text{ and } f_y = \frac{\partial f}{\partial y} = -2y$$

both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

Then we possible (since Theorem 2 does not give necessary and sufficient conditions) have a relative a relative minimum-maximum at $(x,y) = (0,0)$.

Example 9 (2)



Note. The theorem gives necessary but not sufficient conditions

The second derivative test

Suppose that $z = f(x, y)$ has partial derivatives at all points near a point (a, b) and that (a, b) is a critical point of $f(x, y)$ so that

$$f_x(a,b) = 0 = f_y(a,b)$$

Let

$$A = f_{xx}(a,b) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)},$$

$$B = f_{xy}(a,b) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)},$$

$$C = f_{yy}(a,b) = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)}$$

Rules

- **Rule 1.** $f(a,b)$ is a relative maximum if $AC - B^2 > 0$ and $A < 0$.
- **Rule 2.** $f(a,b)$ is a relative minimum if $AC - B^2 > 0$ and $A > 0$.
- **Rule 3.** $\big((a,b), f(a,b)\big)$ is a saddle point if $AC - B^2 < 0$.
- **Rule 4.** The test gives no information about the type of critical point if $AC - B^2 = 0$.
- **Note.** The values of A, B, C depend upon the point (a,b) and must be determined independently for each critical point.

Example 10 (1)

Find the relative minimum-maximum of

$$f(x,y) = x^2 + y^2, -\infty \leq x, y \leq +\infty$$

Since the partial derivatives

$$f_x = \frac{\partial f}{\partial x} = 2x \text{ and } f_y = \frac{\partial f}{\partial y} = 2y$$

both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

$$A = f_{xx}(0,0) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 2 > 0\ ,$$

$$B = f_{xy}(0,0) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = 0,$$

Example 10 (2)

and

$$C = f_{yy}(0,0) = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2$$

$f(0,0) = 0^2 + 0^2 = 0$ is a relative minimum since

$$AC - B^2 = 2 * 2 - 0^2 = 4 > 0$$

and

$$A = 2 > 0$$

Example 11 (1)

Find the relative minimum-maximum of

$$f(x,y) = x^2 - y^2, -\infty \leq x, y \leq +\infty$$

Since the partial derivatives

$$f_x = \frac{\partial f}{\partial x} = 2x \text{ and } f_y = \frac{\partial f}{\partial y} = -2y$$

both exist and

$$f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$$

$$A = f_{xx}(0,0) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 2 > 0\ ,$$

$$B = f_{xy}(0,0) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = 0,$$

Example 11 (2)

and

$$\mathrm{C} = f_{yy}(0,0) = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = -2$$

$((0,0), f(0,0) = 0)$ is a saddle point since

$$AC - B^2 = 2 * (-2) - 0^2 = -4 < 0$$

Absolute Maximum-Minimum

- The **absolute maximum** of the function $f(x, y)$ is a value $f(a, b)$ such that $f(a, b) \geq f(x, y)$ for all values of (x, y) in the domain of $f(x, y)$.
- The **absolute minimum** of the function $f(x, y)$ is a value $f(a, b)$ such that $f(a, b) \leq f(x, y)$ for all values of (x, y) in the domain of $f(x, y)$.

Question: How to find the absolute maximum-minimum of the continuous function $f(x, y)$ on a closed curve R?

How to find the absolute maximum-minimum of the continuous function $f(x, y)$ on a closed curve R?

1. Find all the critical values of $f(x, y)$ i.e.

$$f_x(a, b) = 0 = f_y(a, b)$$

2. Find the maximum of the values of $f(x, y)$ on the boundaries of R.

3. Find the values of $f(x, y)$ for the points where there is no partial derivative of $f(x, y)$.

4. Of the above values, the largest is the absolute maximum, and the smallest is the absolute minimum.

Box optimization (1)

Find the dimensions of a rectangular box (with no top) of volume $64in^3$, that has the minimum surface area.

- **Step 1**. Understand the problem

$V = volume\ of\ the\ box$

$E = surface\ area\ of\ the\ box\ (no\ top)$

$x = the\ length\ of\ the\ first\ side$

$y = the\ length\ of\ the\ second\ side$

$h = the\ heigth\ of\ the\ box$

$V = xyh = 64$

$E = xy + 2xh + 2yh$

Box optimization (2)

- **Step 2.**Form a mathematical statement of the problem.

$$h = \frac{64}{xy}$$

$$E(x,y) = xy + 2x\frac{64}{xy} + 2y\frac{64}{xy} = xy + \frac{128}{y} + \frac{128}{x},$$

$$0 < x, y$$

We seek the maximum value of $E(x,y)$ for $0 < x, y$.

- **Step 3.** Determine the maximum-minimum

Find the derivative of $\mathrm{E}(x,y)$:

$$E_x(x,y) = y - \frac{128}{x^2} = \frac{yx^2 - 128}{x^2}$$

Box optimization (3)

$$E_y(x,y) = x - \frac{128}{y^2} = \frac{xy^2 - 128}{y^2}$$

$$E_y(x,y) = 0 \Rightarrow xy^2 - 128 = 0 \Rightarrow$$

$$x\left(\frac{128}{x^2}\right)^2 - 128 = 0 \Rightarrow 128 = x^3 \Rightarrow$$

$$x = \sqrt[3]{128} = 4\sqrt[3]{2}$$

$$y = \frac{128}{x^2} = \frac{128}{\left(4\sqrt[3]{2}\right)^2} = 4\sqrt[3]{2}$$

Note also that

$$A = E_{xx}\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \frac{2 * 128}{x^3}\bigg|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 2 > 0$$

Box optimization (4)

$$B = E_{xy}\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \frac{\partial^2 E}{\partial y \partial x}\bigg|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 1,$$

$$\text{C}= E_{yy}\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = \frac{2*128}{y^3}\bigg|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 2 > 0$$

$$f\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right) = xy + \frac{128}{y} + \frac{128}{x}\bigg|_{\left(4\sqrt[3]{2},4\sqrt[3]{2}\right)} = 48\sqrt[3]{4}$$

is a relative minimum since

$AC - B^2 = 2 * 2 - 1^2 = 3 > 0$ and $A = 2 > 0$.

Box optimization (5)

- **Step 4.** Answer the question.

Since we have the absolute maximum for $\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right)$, we have that $\mathrm{h} = \frac{128}{\mathrm{xy}} = \frac{128}{4\sqrt[3]{2} * 4\sqrt[3]{2}} = 4\sqrt[3]{2}$ and therefore the dimensions of the box with the maximum surface area are $4\sqrt[3]{2}$ inches by $4\sqrt[3]{2}$ inches by $4\sqrt[3]{2}$ inches.

- **Note.** Find the dimensions of a rectangular box (with no top) with surface area of 64in^2, that has the maximum volume.

Least squares method (1)

Find the line

$$y = ax + b$$

that best fits the data points $(x_i, y_i), i = 1, \dots, n.$

Solution

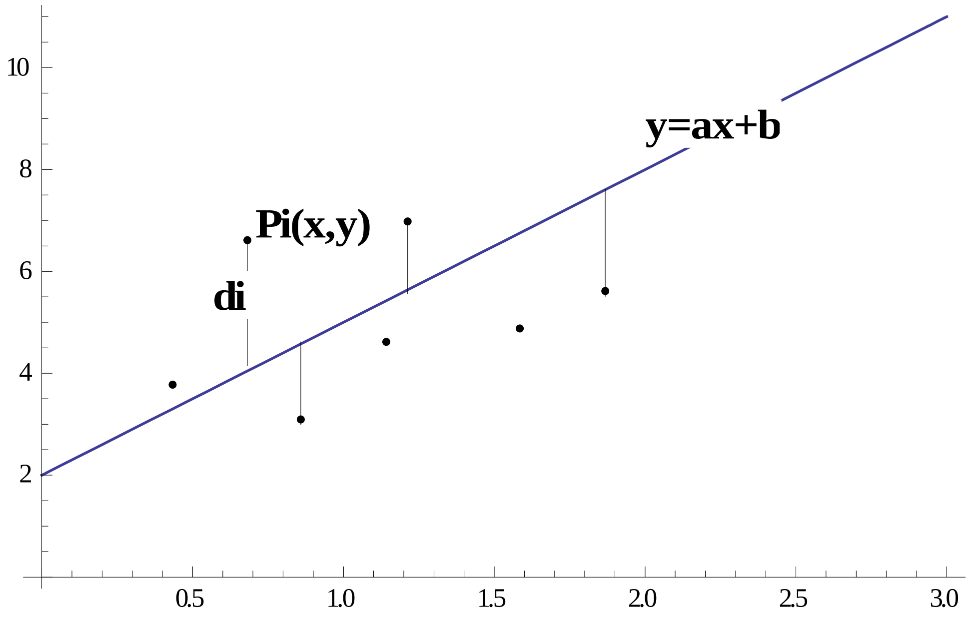
We define the deviation between the ith point and the line to be:

$$d_i = y_i - (ax_i + b)$$

We need to minimize the function

$$f(a,b) = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

Least squares method (2)



Least squares method (3)

$$f_a(a,b) = \frac{\partial}{\partial a}\left\{\sum_{i=1}^{n}(y_i - ax_i - b)^2\right\} =$$

$$= \sum_{i=1}^{n} 2(y_i - ax_i - b)(-x_i) =$$

$$= 2a\sum_{i=1}^{n} {x_i}^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i$$

Least squares method (4)

$$f_b(a,b) = \frac{\partial}{\partial b}\left\{\sum_{i=1}^{n}(y_i - ax_i - b)^2\right\} =$$

$$= \sum_{i=1}^{n} 2(y_i - ax_i - b)(-1) =$$

$$= 2a\sum_{i=1}^{n} x_i + 2b\sum_{i=1}^{n} 1 - 2\sum_{i=1}^{n} y_i$$

Solve

$$f_a(a,b) = f_b(a,b) = 0$$

or equivalently

Least squares method (5)

$$a\sum_{i=1}^{n} {x_i}^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

or

$$\begin{pmatrix} \sum_{i=1}^{n} {x_i}^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix} \binom{a}{b} = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix} \Rightarrow$$

Least squares method (6)

$$\binom{a}{b} = \frac{1}{\left(n\sum_{i=1}^{n} x_i{}^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right)} \begin{pmatrix} n & -\sum_{i=1}^{n} x_i \\ -\sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i{}^2 \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

$\Rightarrow$

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i{}^2 - \left(\sum_{i=1}^{n} x_i\right)^2},$$

$$b = \frac{-\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\, y_i\right) + \left(\sum_{i=1}^{n} x_i{}^2\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i{}^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Note. Use the second derivative test in order to show that we have a minimum for these (a,b).

Constrained Optimization Problems and Lagrange Multipliers (1)

Minimize-Maximize

$$Q(x, y)$$

subject to constraints $C(x, y) = 0$.

Define the Lagrange function as

$$L(x, y, \lambda) = Q(x, y) + \lambda C(x, y)$$

Then all the relative minimum and maximum points of $Q(x, y)$ with x and y constrained to satisfy the equation $C(x, y) = 0$ will be among those points (x_0, y_0) for which (x_0, y_0, λ_0) is a maximum or minimum point of $L(x, y, \lambda)$.

Constrained Optimization Problems and Lagrange Multipliers (2)

These points (x_0, y_0, λ_0) will be solutions of the system of simultaneous equations

$$L_x(x, y, \lambda) = 0$$

$$L_y(x, y, \lambda) = 0$$

$$L_\lambda(x, y, \lambda) = 0 \text{(this is just } C(x, y) = 0\text{)}$$

We assume that all partial derivatives exist.

Example 12 (1)

Minimize

$$E = (2x) * (2y) = 4xy$$

subject to constraints $x^2 + y^2 = 20^2$.

Define the Lagrange function

$$L(x, y, \lambda) = 4xy + \lambda(x^2 + y^2 - 20^2)$$

Find the points that satisfy the system of simultaneous equations

$$L_x(x, y, \lambda) = 4y + 2\lambda x = 0$$

$$L_y(x, y, \lambda) = 4x + 2\lambda y = 0$$

$$L_\lambda(x, y, \lambda) = x^2 + y^2 - 20^2 = 0$$

or

Example 12 (2)

equivalently

$$y = -\frac{1}{2}\lambda x$$

$$x = -\frac{1}{2}\lambda y \Rightarrow x = -\frac{1}{2}\lambda\left(-\frac{1}{2}\lambda x\right) = \frac{1}{4}\lambda^2 x \Rightarrow \left(1 - \frac{\lambda^2}{4}\right)x = 0$$

$\Rightarrow x = 0 \vee \lambda = \pm 2$

For $x = 0$ we have that $\mathrm{y} = 0$ but $0^2 + 0^2 \neq 20^2$. Therefore $x = -y$ for $\lambda = 2$ (or $y = x$ for $\lambda = -2$) and

$$x^2 + y^2 - 20^2 = 0 \Rightarrow x^2 + (-x)^2 - 20^2 = 0 \Rightarrow$$

$$2x^2 = 20^2 \Rightarrow$$

$$x = \pm 10\sqrt{2}$$

Example 12 (3)

and thus

$$y = -x = \mp 10\sqrt{2}$$

(respectively $y = x = \pm 10\sqrt{2}$)

Thus the only candidates for minimum-maximum points of the constrained optimization problem are the pairs $\left(\pm 10\sqrt{2}, \pm 10\sqrt{2}\right)$and $\left(\pm 10\sqrt{2}, \mp 10\sqrt{2}\right)$.

The values of the function $E = (2x) * (2y) = 4xy$ at these points are respectively

$$E = \left(\pm 10\sqrt{2}, \pm 10\sqrt{2}\right) = 800$$

and

Example 12 (4)

$$E = \left(\pm 10\sqrt{2}, \mp 10\sqrt{2}\right) = -800$$

Therefore we have a minimum for the pair

$\left(10\sqrt{2}, -10\sqrt{2}\right)$ (and $\left(-10\sqrt{2}, 10\sqrt{2}\right)$)

and maximum for the pair

$\left(10\sqrt{2}, 10\sqrt{2}\right)$ (and $\left(-10\sqrt{2}, -10\sqrt{2}\right)$).

Preliminaries in Matrix Theory Positive Definite Matrices

Let P, S be real symmetric matrices. If

$$y^T P y > 0 \ \forall y \neq 0$$

then P is called positive-definite matrix. If

$$y^T P y \geq 0 \ \forall y \neq 0$$

then S is called positive semi-definite matrix.

Notes.

- If P is positive definite matrix then P is invertible.
- If P is positive definite and S is positive semi-definite then $P + S$ is positive definite.
- The real symmetric matrix P is positive definite (semi-definite) iff the eigenvalues of P are positive (non negative)

Negative Definite Matrices

Let P, S be real symmetric matrices. If

$$y^T P y < 0 \ \forall y \neq 0$$

then P is called negative-definite matrix. If

$$y^T P y \leq 0 \ \forall y \neq 0$$

then S is called negative semi-definite matrix.

Notes.

- If P is negative definite matrix then P is invertible.
- If P is negative definite and S is negative semi-definite then $P + S$ is negative definite.
- The real symmetric matrix P is negative definite (semi-definite) iff the eigenvalues of P are negative (non positive).

Positive - Negative Definite Matrices

The matrix P

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

is positive definite iff

$$p_{11} > 0, \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} > 0, \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix} > 0$$

and negative definite iff

$$p_{11} < 0, \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} > 0, \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix} < 0 \text{(sign changes)}$$

Positive - Negative Semi-definite Matrices

The matrix P

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

is positive semi-definite iff

$$p_{11} \geq 0, \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} \geq 0, \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix} \geq 0$$

and negative semi-definite iff

$$p_{11} \leq 0, \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} \geq 0, \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix} \leq 0 (\text{sign changes})$$

Gradient of a function

Consider the function $f(x_1, x_2, \dots, x_n)$.

We define the gradient of $f\colon \mathbb{R}^n \to \mathbb{R}$ with respect to x as

$$\nabla_x f(x) = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

where $\mathrm{x} = (x_1, x_2, \dots, x_n)^T$.

Jacobian Matrix

The total differential of a function $f: \mathbb{R}^m \to \mathbb{R}^n$ is defined by the Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & & \frac{\partial f_n}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_m} & \frac{\partial f_2}{\partial x_m} & & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

Properties of the Gradient Functions (1)

1. $\frac{\partial(x^T c)}{\partial x} = \frac{\partial(x_1c_1+x_2c_2+\cdots+x_mc_m)}{\partial x} = c$

2. $\frac{\partial(Ax)}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix} =$

$$= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{mn} \end{bmatrix} = A^T$$

3. $\frac{\partial(x^T A)}{\partial x} = A$

Properties of the Gradient Functions (2)

4. $$\frac{\partial\left(x^T M x\right)}{\partial x} = \frac{\partial\left(x^T \underbrace{[Mx]}_{c_1}\right)}{\partial x} + \frac{\partial\left(\left(\underbrace{\left[M^T x\right]}_{c_2}^T\right) x\right)}{\partial x} =$$

$$= \frac{\partial\left(x^T \underbrace{[Mx]}_{c_1}\right)}{\partial x} + \frac{\partial\left(x^T \underbrace{\left[M^T x\right]}_{c_2}\right)}{\partial x} = Mx + M^T x$$

If is M real symmetric matrix then $\frac{\partial\left(x^T M x\right)}{\partial x} = 2Mx$.

Hessian Matrix of Functions

We define the Hessian matrix of $f(x)$ as

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \left[\frac{\partial^2 f}{\partial x \partial x}\right] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_m} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots \frac{\partial^2 f}{\partial x_2 \partial x_m} \\ \vdots & \vdots & \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \frac{\partial^2 f}{\partial x_m \partial x_2} & \frac{\partial^2 f}{\partial x_m^2} \end{bmatrix}$$

Properties of the Hessian Matrix

Properties

1. $\frac{\partial^2(x^T M x)}{\partial x^2} = \frac{\partial\left((M+M^T)x\right)}{\partial x} = \frac{\partial\left(x^T(M^T+M)\right)}{\partial x} = M + M^T.$
2. If is a real symmetric matrix then $\frac{\partial^2(x^T M x)}{\partial x^2} = 2M.$
3. $f = f\left(x(t), y(t)\right) \Rightarrow df = \left[\frac{\partial f}{\partial x}\right]^T dx + \left[\frac{\partial f}{\partial y}\right]^T dy$
4. $f = f(x(t), y(t), t),\ y(t) = y(x(t), t\ )$

- $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial y^T}{\partial x}\frac{\partial f}{\partial y}$
- $\frac{\partial f}{\partial t} = \left[\frac{\partial f}{\partial x} + \frac{\partial y^T}{\partial x}\frac{\partial f}{\partial y}\right]^T \frac{\partial x}{\partial t} + \left[\frac{\partial f}{\partial y}\right]^T \frac{\partial y}{\partial t} + \frac{\partial f}{\partial t}$

Taylor series of a function

$$f(x) = f(x_0) + \left[\frac{\partial f}{\partial x}\right]^T\Bigg|_{x=x_0} (x - x_0) + \frac{1}{2!}(x - x_0)^T \left[\frac{\partial^2 f}{\partial x^2}\right]\Bigg|_{x=x_0} (x - x_0) + O(3)$$

$$\Delta f(x) = \left[\frac{\partial f}{\partial x}\right]^T\Bigg|_{x=x_0} dx + \frac{1}{2!} dx^T \left[\frac{\partial^2 f}{\partial x^2}\right]\Bigg|_{x=x_0} dx + O(3)$$

Relative Maxima-Minima (1)

- A function $f: D \to \mathbb{R}, D \subseteq \mathbb{R}^n$ has a **relative maximum** $f(a)$ at the point $(a, f(a))$ if $f(a) \geq f(x)$ for all x in some region containing a.
- A function $f: D \to \mathbb{R}, D \subseteq \mathbb{R}^n$ has a **relative minimum** $f(a)$ at the point $(a, f(a))$ if $f(a) \leq f(x)$ for all x in some region containing a.

Theorem 3. (Necessary Conditions) Suppose that $f(x)$ attains a relative minimum value (resp. relative maximum) at the point a and the gradient $\nabla_x f$ exists. Then

1) $\nabla_x f|_{x=a} = 0 \Leftrightarrow \left.\frac{\partial f}{\partial x_i}\right|_{x_i=a_i} = 0$

Relative Maxima-Minima (2)

2) $K = [k_{ij}] = \left[\frac{\partial^2 f(x)}{\partial x^2}\right]\Big|_{x=a} = f_{xx}|_{x=a}$ is a positive semi-definite (resp. negative semi-definite)

Sufficient Condition: K is positive definite (resp. negative definite)

Notes:

- If $x^T K x$ change signs at $x = a$ then a is a saddle point.
- If K is positive semi-definite or negative semi-definite then we need more information in order to decide if the point is minimum or maximum (terms of order 3). The point a is called singular.

Example 13 (1)

Find the relative minimum-maximum of

$$f(x, y, z) = x^2 + y^2 + z^2, -\infty \leq x, y, z \leq +\infty$$

Since the partial derivatives

$$f_x = \frac{\partial f}{\partial x} = 2x, f_y = \frac{\partial f}{\partial y} = 2y, f_z = \frac{\partial f}{\partial z} = 2z$$

exist and

1. $f_x = f_y = f_z = 0 \Leftrightarrow (x, y, z) = (0,0,0)$
2. $K = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} > 0$

Example 13 (2)

Thus we have a relative minimum at $(x, y, z) = (0,0,0)$

$$f(x, y, z) - f(0,0,0) = x^2 + y^2 + z^2 - 0 =$$

$$= x^2 + y^2 + z^2 \geq 0 \Rightarrow$$

$$f(x, y, z) - f(0,0,0) \geq 0 \Rightarrow$$

$$f(x, y, z) \geq f(0,0,0)$$

Example 14 (1)

Find the relative minimum-maximum of

$$f(x,y) = x^2 - y^2, -\infty \leq x, y \leq +\infty$$

Since the partial derivatives

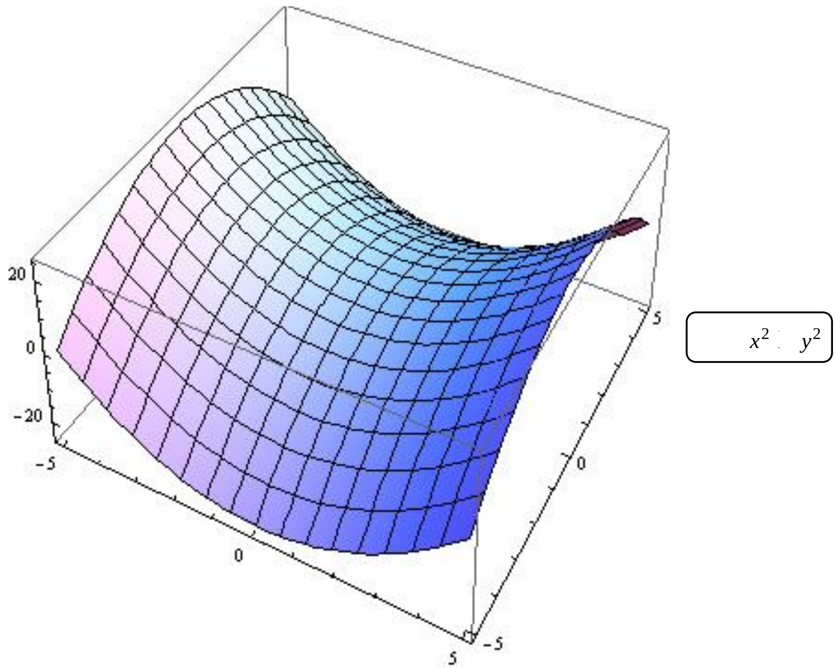
$$f_x = \frac{\partial f}{\partial x} = 2x, f_y = \frac{\partial f}{\partial y} = -2y$$

both exist and

1. $f_x = f_y = 0 \Leftrightarrow (x,y) = (0,0)$
2. $K = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

Example 14 (2)

We have a saddle point at $(x, y) = (0,0)$ since K has positive and negative eigenvalues.



Constrained Optimization

Let $J(x,u): \mathbb{R}^n \to \mathbb{R}$ and $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x,u) = 0$ where $\mathrm{u}(\mathrm{t}) \in \mathbb{R}^r, x(t) \in \mathbb{R}^n$.

Problem. Find the minimum-maximum of $J(x,u)$ under the constraints $f(x,u) = 0$.

1st solution. Solve the second equation and substitute in the function $J(x,u)$. Then apply the known criteria.

2nd solution. Lagrange multipliers.

From the function

$$L(x,u,\lambda) = J(x,u) + \lambda f(x,u)$$

$\lambda = (\lambda_1, \lambda_2, .., \lambda_m), \lambda_i: Lagrange\ multipliers$

$L(x,u,\lambda)$: Lagrangian

Necessary Conditions

Then all the relative minimum and maximum points of $J(x,u)$, with x and y constrained to satisfy the equation $f(x,u) = 0$, will be among those points

$$(x_0, u_0)$$

for which (x_0, u_0, λ_0) is a maximum or minimum point of $L(x,u,\lambda)$.

These points (x_0, u_0, λ_0) will be the solutions of the system of simultaneous equations

$$L_x(x,y,\lambda) = 0$$

$$L_y(x,y,\lambda) = 0$$

$$L_\lambda(x,y,\lambda) = 0 \text{(this is just } f(x,y) = 0\text{)}$$

Example 15 (1)

Find the point on the plane

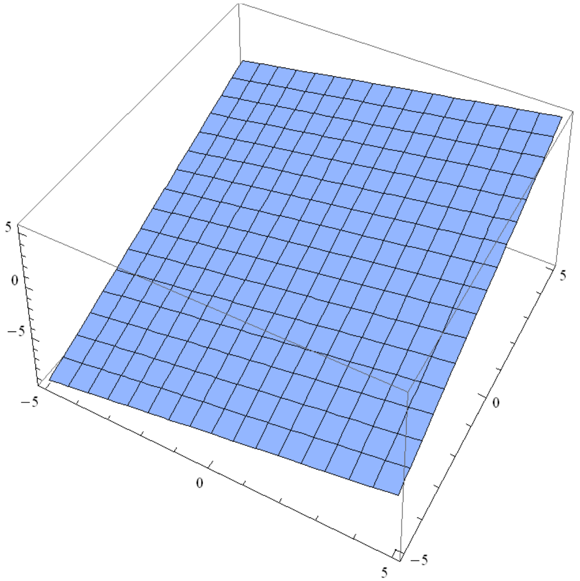
$$x + 2y + 2z = 4$$

that is closest to the origin or equivalently minimize

$$f(x, y, z) = x^2 + y^2 + z^2$$

Subject to constraint

$$x + 2y + 2z = 4$$



Example 15 (2)

Define the Lagrangian

$$L(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(x + 2y + 2z - 4\,)$$

The necessary conditions are

$$\left.\begin{array}{r} \dfrac{\partial L}{\partial x} = 2x + \lambda = 0 \\ \dfrac{\partial L}{\partial y} = 2y + 2\lambda = 0 \\ \dfrac{\partial L}{\partial z} = 2z + 2\lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = x + 2y + 2z - 4 = 0 \end{array}\right\} \Rightarrow \left\{x = \frac{4}{9}, y = \frac{8}{9}, z = \frac{8}{9}, l = -\frac{8}{9}\right\}$$

Sufficient Conditions

Let

$$D_\varphi(x) = \begin{bmatrix} \nabla_{\varphi_1}^{\mathrm{T}} \\ \nabla_{\varphi_2}^{\mathrm{T}} \\ \vdots \\ \nabla_{\varphi_m}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial f}{\partial x}\right)^T & \left(\dfrac{\partial f}{\partial u}\right)^T \end{bmatrix}$$

the Jacobian of the constraints

and

$$T_\varphi(x) = \{\xi : D_\varphi(x)\xi = 0\}$$

be the tangent plane at the point x on the surface defined by the constraints. Then

Theorem 3

Suppose

$$f(x), \varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)$$

have continuous second partial derivatives in $\mathbb{R}^n$ and let

$$(x^*, \lambda^*)$$

be a stationary point of the Lagrangian $L(x, \lambda)$.

If

$$\xi^T L_{xx}(x^*, \lambda^*)\xi > 0, \forall \xi (\neq 0) \in T_\varphi(x^*)$$

then x^* is a strong local minimiser of $f(x)$ subject to constraints

$$\varphi_1(x) = \varphi_2(x) = \cdots = \varphi_m(x) = 0.$$

Example 15 (3)

$$T_\varphi(x) = \{\xi : D_\varphi(x)\xi = 0\} = \left\{\xi : \left[\left(\frac{\partial f}{\partial x}\right)^T \quad \left(\frac{\partial f}{\partial u}\right)^T\right]\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = 0\right\}$$

$$= \begin{bmatrix} -\left(\frac{\partial f}{\partial x}\right)^{-T}\left(\frac{\partial f}{\partial u}\right)^T \xi_2 \\ \xi_2 \end{bmatrix}$$

$$\xi^T L_{xx}(x^*, \lambda^*)\xi =$$

$$= \xi_2^{\mathrm{T}} \underbrace{\left[-\left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial x}\right)^{-1} \quad \mathrm{I}\right] L_{xx}(x^*, \lambda^*) \begin{bmatrix} -\left(\frac{\partial f}{\partial x}\right)^{-T}\left(\frac{\partial f}{\partial u}\right)^T \\ I \end{bmatrix}}_{J_{uu}^f} \xi_2 > 0$$

$$\forall \xi_2 (\neq 0)$$

Example 15 (4)

$$L(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(x + 2y + 2z - 4\ )$$

$$D_\varphi(x) = \nabla_{\varphi_1}^{\mathrm{T}} = (1 \quad 2 \quad 2)$$

$$T_\varphi(x) = \{\xi\colon (1 \quad 2 \quad 2)\xi = 0\} = \begin{bmatrix} -2\xi_2 - 2\xi_3 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$L_{xx} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} > 0$$

$$\xi^T L_{xx}(x^*, \lambda^*)\xi$$

$$= [-2\xi_2 - 2\xi_3 \quad \xi_2 \quad \xi_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2\xi_2 - 2\xi_3 \\ \xi_2 \\ \xi_3 \end{bmatrix} =$$

Example 15 (5)

$$= 2(-\xi_2 - \xi_3)^2 + 2(\xi_2)^2 + 2(\xi_3)^2 > 0, \forall \xi \neq 0$$

Therefore

$$\left\{x = \frac{4}{9}, y = \frac{8}{9}, z = \frac{8}{9}, l = -\frac{8}{9}\right\}$$

minimize the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to constraint

$$x + 2y + 2z = 4$$

Βιβλιογραφία

- Ron Larson, Bruce H. Edwards, 2013, Calculus, Cengage Learning (10 edition)

Σημείωμα Αναφοράς

 «Θεωρία Βέλτιστου Ελέγχου. **Ενότητα 3**: Ακρότατα συναρτήσεων μίας ή πολλών μεταβλητών». Έκδοση: 1.0. Θεσσαλονίκη 2014.

Διαθέσιμο από τη δικτυακή διεύθυνση: http://eclass.auth.gr/courses/OCRS288/

Σημείωμα Αδειοδότησης

Διατήρηση Σημειωμάτων

Οποιαδήποτε αναπαραγωγή ή διασκευή του υλικού θα πρέπει να συμπεριλαμβάνει:

- το Σημείωμα Αναφοράς
- το Σημείωμα Αδειοδότησης
- τη δήλωση Διατήρησης Σημειωμάτων
- το Σημείωμα Χρήσης Έργων Τρίτων (εφόσον υπάρχει)

μαζί με τους συνοδευόμενους υπερσυνδέσμους.

ΑΡΙΣΤΟΤΕΛΕΙΟ
ΠΑΝΕΠΙΣΤΗΜΙΟ
ΘΕΣΣΑΛΟΝΙΚΗΣ

ΑΝΟΙΧΤΑ
ΑΚΑΔΗΜΑΙΚΑ
ΜΑΘΗΜΑΤΑ

Τέλος Ενότητας

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Θεσσαλονίκη, Εαρινό εξάμηνο 2013-2014

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