Aristotle University
of Thessaloniki

Academic
Open Courses

Groundwater Hydraulics

Unit 3: The method of images

K.L. Katsifarakis

Department of Civil Engineering Aristotle University of Thessaloniki,

Macedonia, Greece

Usage Rights

A short presentation of the method of images

The method of images offers exact analytical solutions in cases of flow fields with one straight-line boundary. It is actually a procedure to pass to an equivalent infinite flow field, where we can use the respective formulas for well systems.

The cost that we have to accept is that the equivalent infinite field bears a number of additional imaginary wells. If there are N wells in the real flow field, which is confined by a rectilinear flow boundary, the equivalent infinite field will have 2N wells.

The imaginary wells are symmetric to the real ones, with respect to the flow boundary. The name of the method is due to this fact.

If we use the method of images, in order to "get rid" of a constant head boundary, flow rates of the imaginary wells have opposite sign of those of the real wells, namely the images of pumped wells are injection wells.

If we consider that y-axis coincides with the constant head boundary, the following equation holds:

$$s = -\frac{1}{2\pi K\alpha}\sum_{i=1}^{n} Q_i \cdot \ln\frac{\sqrt{(x-x_i)^2+(y-y_i)^2}}{\sqrt{(x+x_i)^2+(y-y_i)^2}}$$

This formula guarantees that hydraulic head drawdown along the y-axis is equal to zero, namely that the boundary condition of the semi-infinite field is observed.

If an impermeable boundary exists, the imaginary wells have the same sign as the real ones.

If we consider that y-axis coincides with the impermeable boundary, the following equation holds:

$$s = -\frac{1}{2\pi K\alpha}\sum_{i=1}^{n} Q_i \cdot \ln \frac{\sqrt{(x-x_i)^2+(y-y_i)^2}\cdot\sqrt{(x+x_i)^2+(y-y_i)^2}}{R^2}$$

This formula guarantees that along the y-axis, velocities perpendicular to it are equal to zero, namely that the boundary condition of the semi-infinite field is observed.

For free surface flows and a constant head boundary we get:

$$H^2 = h_1^2 + \frac{1}{\pi K}\sum_{i=1}^{n} Q_i \cdot \ln \frac{\sqrt{(x-x_i)^2 + (y-y_i)^2}}{\sqrt{(x+x_i)^2 + (y-y_i)^2}}$$

For n wells close to an impermeable boundary we have:

$$H^2 = h_1^2 + \frac{1}{\pi K}\sum_{i=1}^{n} Q_i \cdot \ln \frac{\sqrt{(x-x_i)^2 + (y-y_i)^2} \cdot \sqrt{(x+x_i)^2 + (y-y_i)^2}}{R^2}$$

Mathematical and physical reasoning

From the mathematical point of view, addition of imaginary wells aims at observing the boundary condition. For constant head boundaries, hydraulic head level drawdown should be zero ($s = 0$). Each real well that pumps (or injects) a flow rate equal to Q_i, results in a hydraulic head level change s_i. This change is balanced, if an imaginary well, symmetric to the real one with respect to the boundary, has a flow rate equal to $-Q_i$.

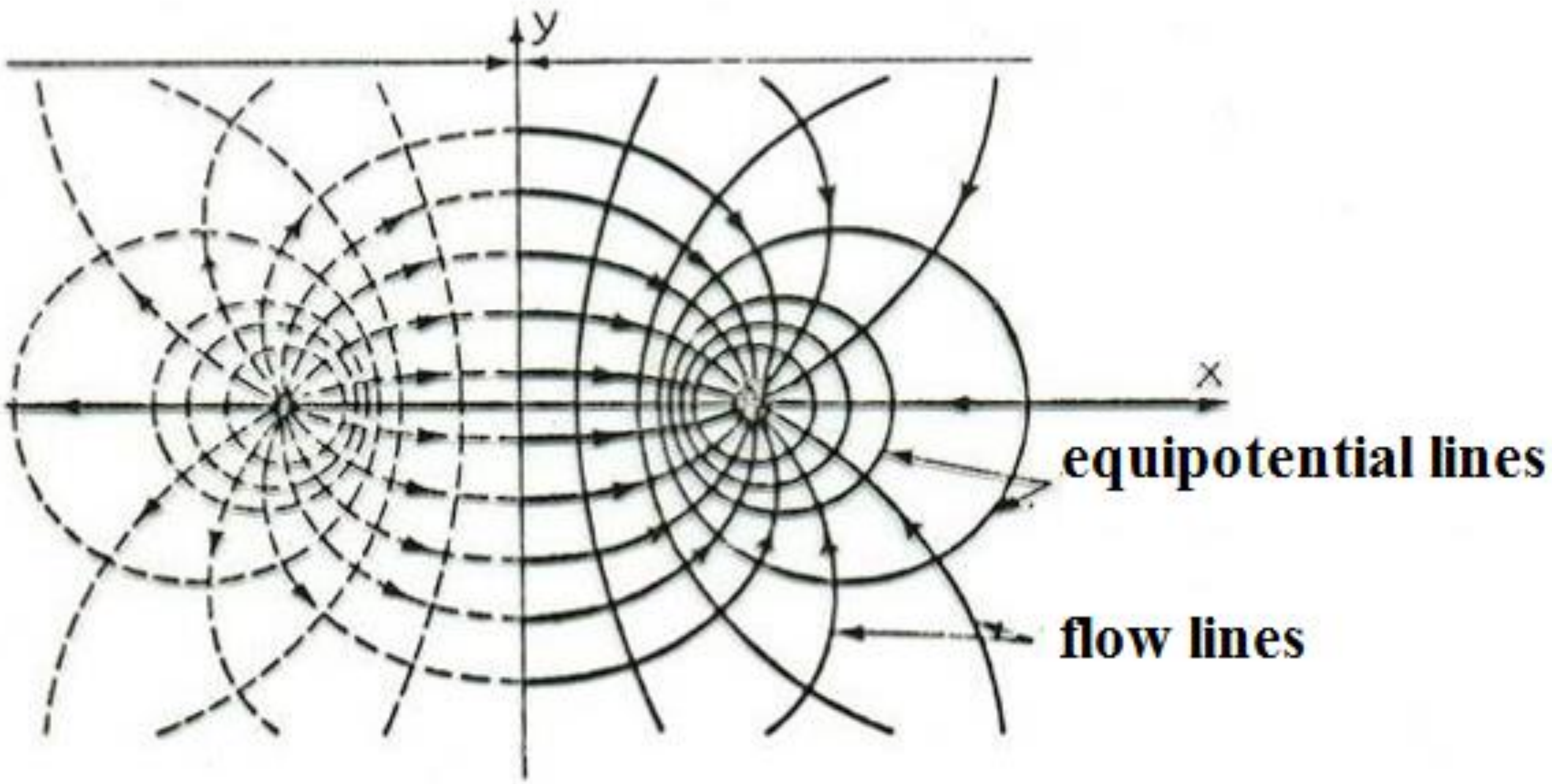
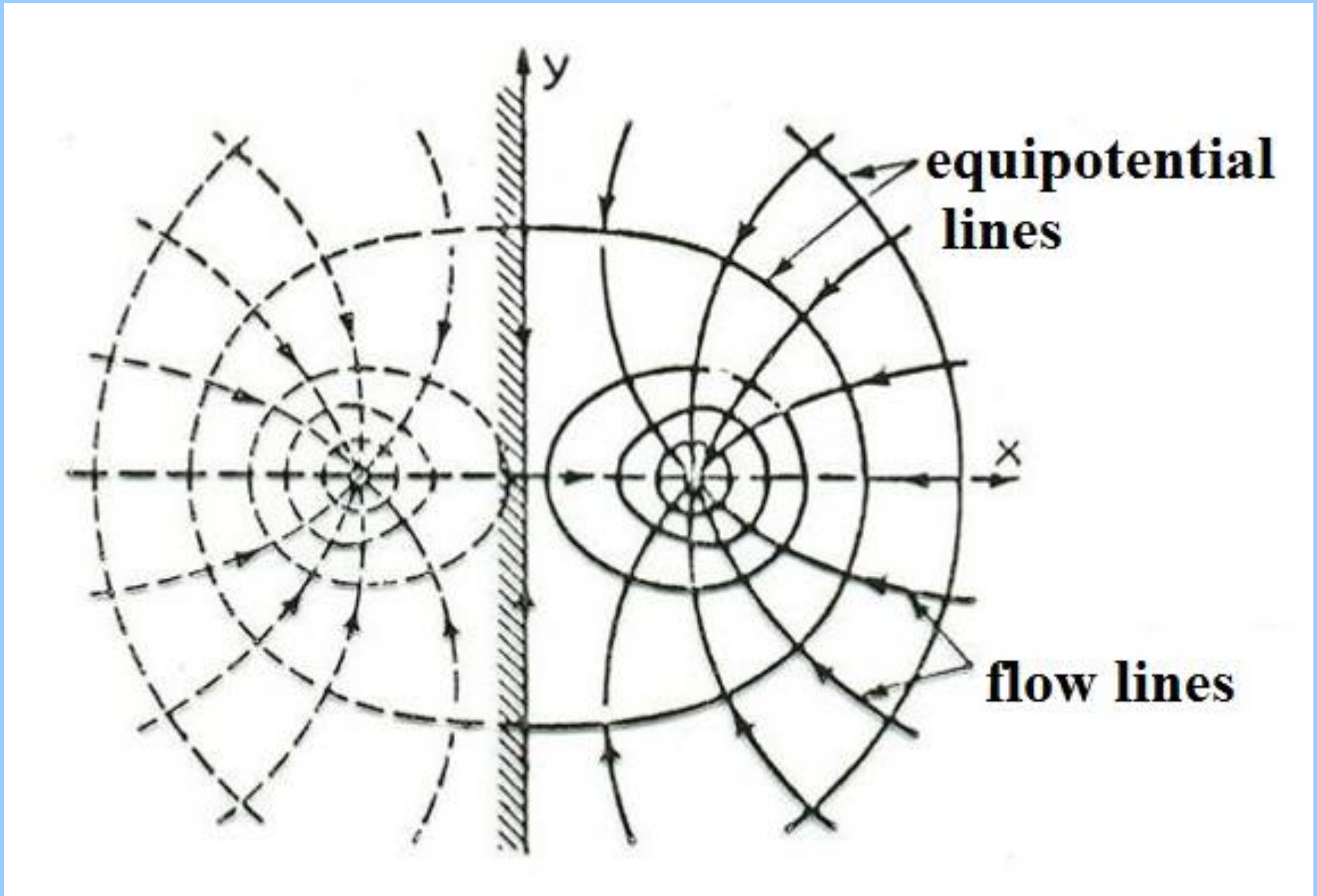


Figure adapted from: https://opencourses.auth.gr/courses/OCRS179/

At any point of an impermeable boundary, groundwater velocity vertical to it should be zero. At boundary point j, the velocity induced by well i has the direction ij. Its component V_n, vertical to the boundary, is counterbalanced, if an imaginary well, symmetric to the real one with respect to the boundary, pumps equal flow rate Q_i.



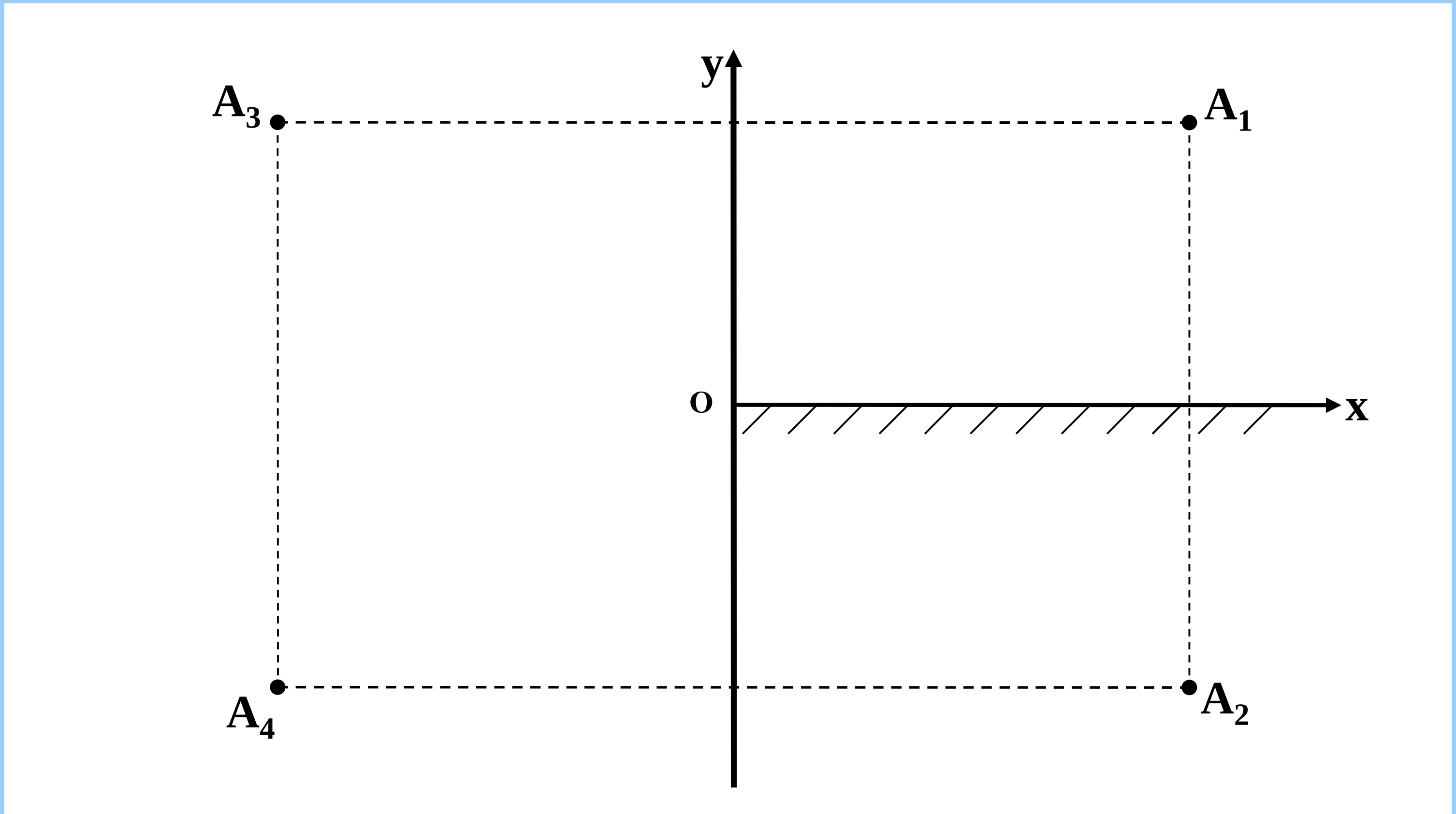
Physical reasoning

The impermeable boundary induces a water deficit, with respect to the infinite field. It is reasonable then to use pumping wells as images of pumping wells, in order to “pump” the water that would otherwise enter the real flow field from the area of the impermeable boundary.

On the contrary, constant head boundary (which could represent the bank of a lake, a river or the sea) induces a water surplus, with respect to the infinite aquifer. It is reasonable then to have injection wells as images of pumping wells, in order to offer the additional water quantity.

Aquifers with more than one boundaries

The method of images offers exact solutions, in some cases of semi-infinite aquifers, confined by two intersecting straight boundaries. The number of the required imaginary wells increases, as the angle formed by the two boundaries decreases.



For an exact solution to exist, the following conditions should hold: a) The number of image wells should be finite. b) Their kind (pumping or inhecting) should be uniquely defined and c) Image wells should not be placed inside the real flow field.

An exact solution exists when the angle of intersection of the two boundaries is an integer submultiple of 180º or 90º, if the boundaries are of the same or of different kind respectively. The number m of the imaginary wells is given as:

$$m = \frac{360}{\theta} - 1$$

where θ is the angle of intersection of the two boundaries (in degrees). It can be easily proved that all the imaginary wells (and the real one) lie on a circle. Its center is the intersection point O of the two boundaries, while its radius is equal to the distance of O from the real well.

The method of images applies also when the flow boundary is an interface between two zones of different transmissivities T_1 and T_2. If the well, with coordinates (x_1,y_1), is inside zone 1, then the hydraulic head level drawdown at any point (x,y) of the same zone is given as:

$$s = -\frac{Q_1}{2\pi T_1}\left(\ln \frac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{R} + \frac{T_1-T_2}{T_1+T_2}\cdot \ln \frac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{R} \right)$$

If point (x,y) belongs to zone 2, then the hydraulic head level drawdown is given as:

$$s = -\frac{Q_1}{\pi(T_1+T_2)} \ln \frac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{R}$$

Actually, the constant head boundary and the impermeable boundary could be considered as limiting cases of the interface boundary.

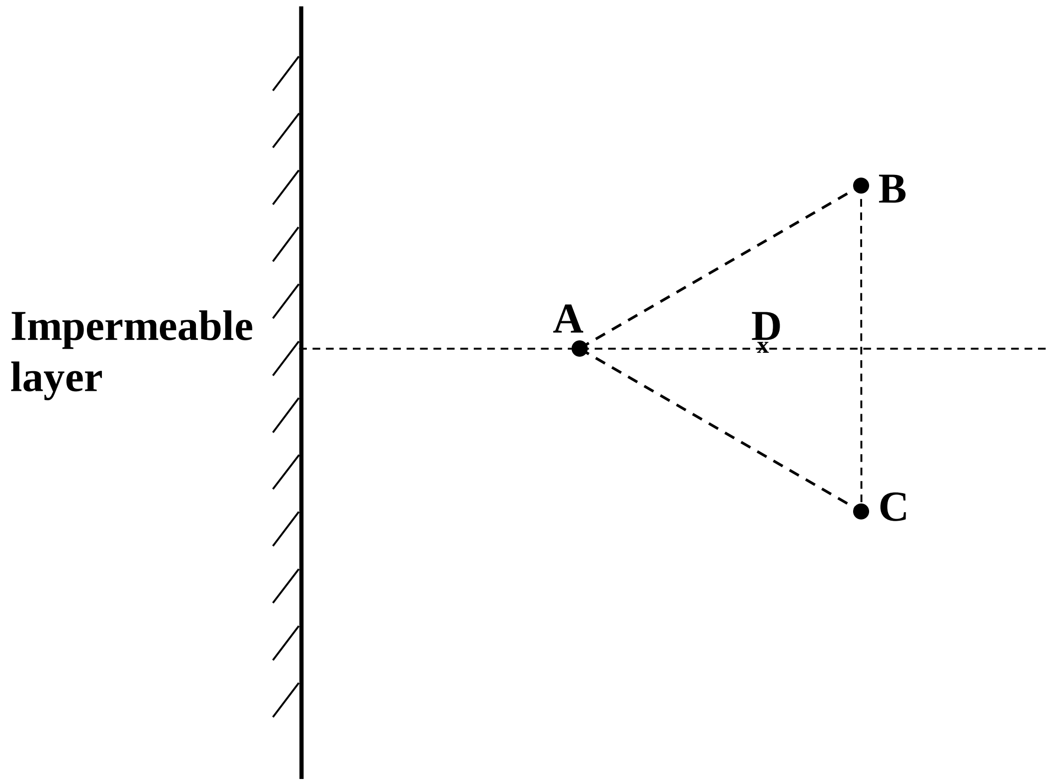
<u>Exercise 1</u>: Wells A, B and C, which form an equilateral triangle, pump water from a semi-infinite aquifer.

Data: a) Well radii are equal

b) Well flow rates are equal

c) The flow is confined

d) Distances between wells and between wells and the boundary are much smaller than the radius of influence of the system of wells.



e) Hydraulic head level drawdown at A is s_A = 53 m.

Hydraulic head level drawdown s_B at B has one of the following values: 49, 51, 53, 55

Hydraulic head level drawdown s_C at C has one of the following values: 51, 53, 55, 57

Hydraulic head level drawdown s_D at D, pericentre of triangle ABC, has one of the following values: 43, 51, 53, 55, 63

Define s_B, s_C and s_D

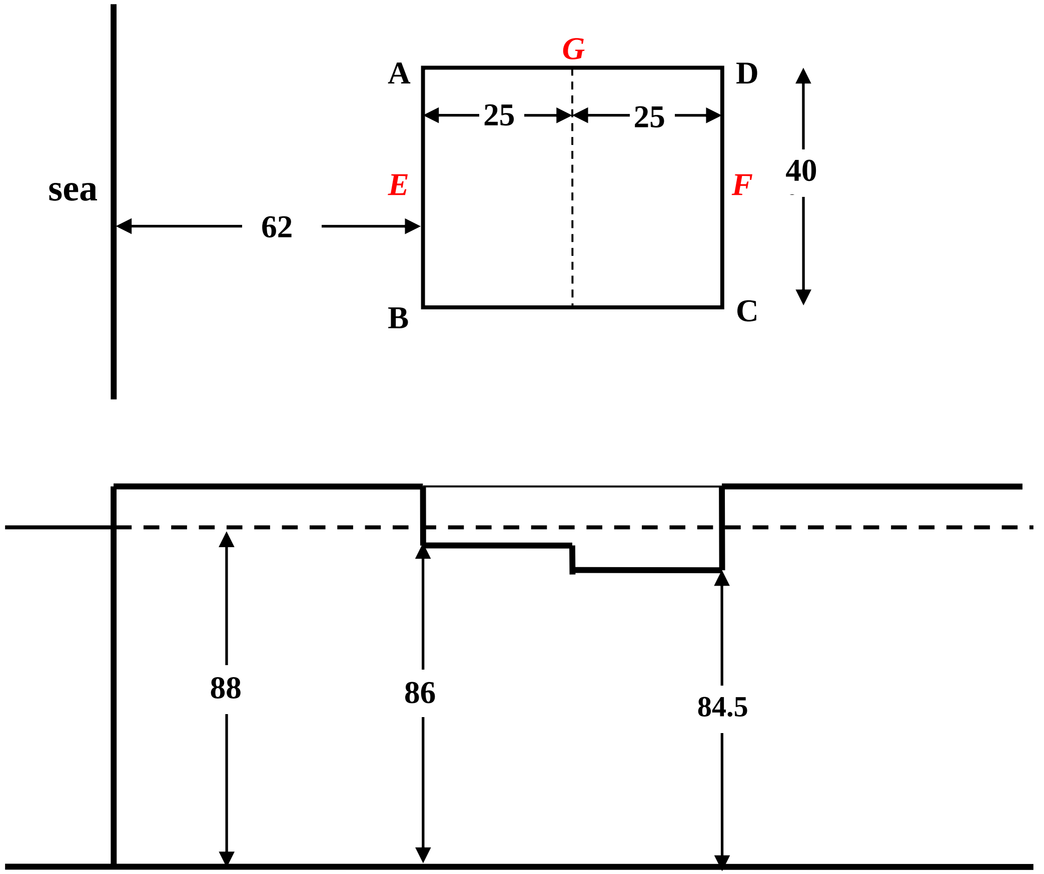
Exercise 2

The bottom of the rectangular excavation ABCD, shown in the figure, is flooded by seawater. To render it dry, a well will be constructed at a point of side AB or at a point of side CD.

a) Which point should be selected, in order to do the prescribed job by pumping the minimum flow rate?

b) Calculate the value of the minimum required pumping flow rate.

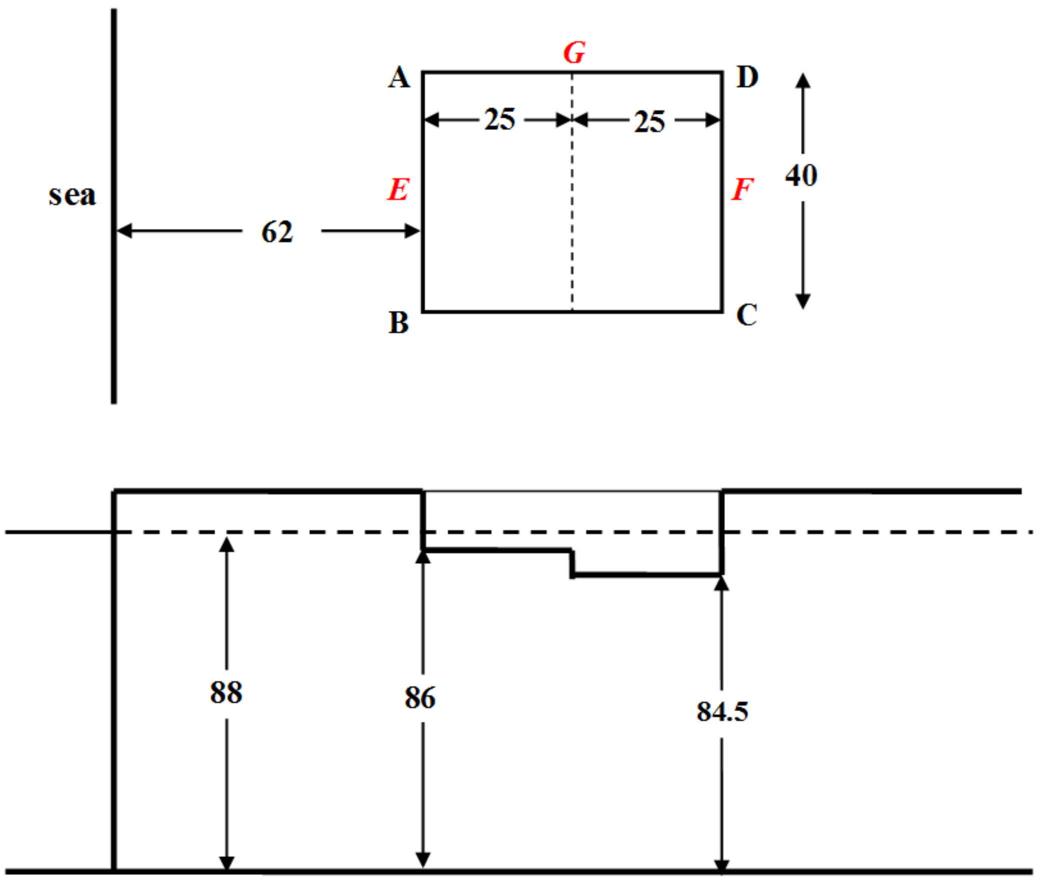
Data: Hydraulic conductivity $K = 4.2 \cdot 10^{-5}$ m/s.



Solution

The method of images should be used, due to constant head boundary.

The most suitable point of side AB is its midpoint E.



Check at D:

$$H^2 = h_1^2 + \frac{Q}{\pi K}\ln\frac{\sqrt{50^2+20^2}}{\sqrt{174^2+20^2}} \Rightarrow$$

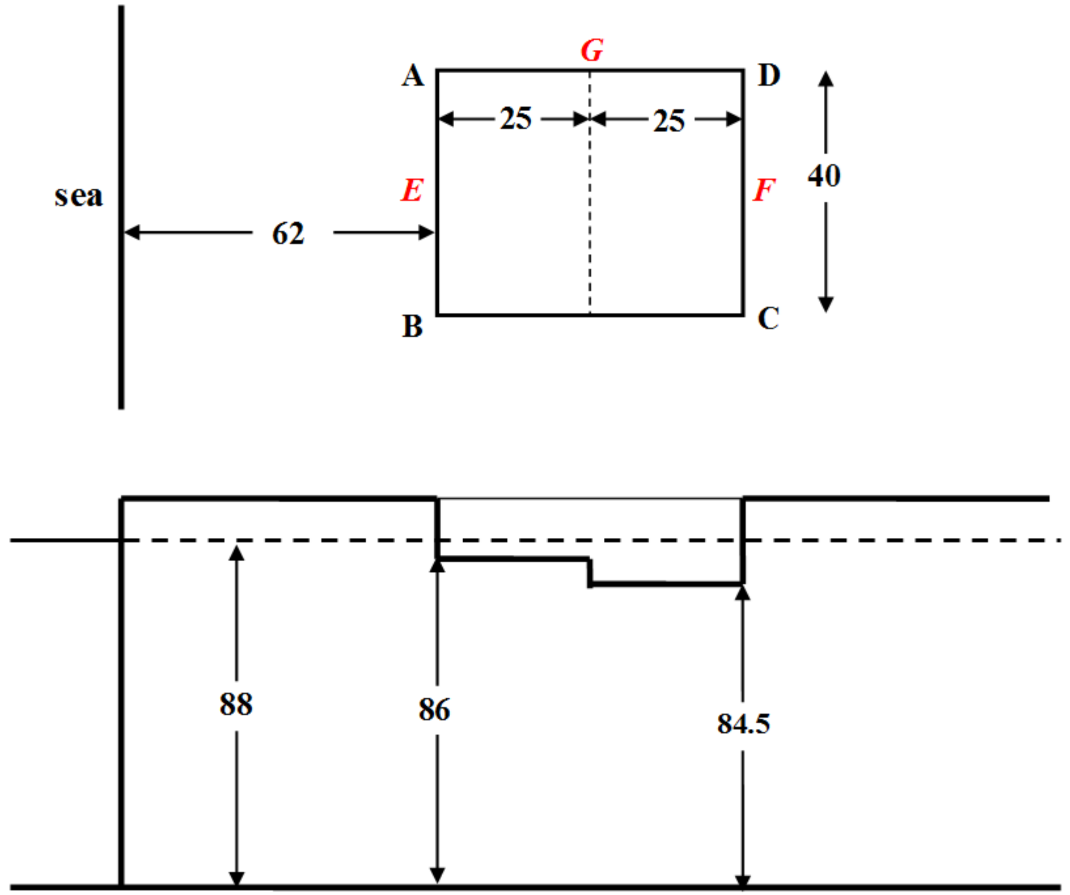
$$\Rightarrow Q = 0.0675 m^3/s$$

Check at G:

$$H^2 = h_1^2 + \frac{Q}{\pi K}\ln\frac{\sqrt{25^2+20^2}}{\sqrt{149^2+20^2}} \Rightarrow Q = 0.0515\ m^3/s$$

Check at A: $$H^2 = h_1^2 + \frac{Q}{\pi K}\ln\frac{20}{\sqrt{124^2+20^2}} \Rightarrow Q = 0.025\ m^3/s$$

The most suitable point of side CD is its midpoint F, since it has the minimum distance of the respective control points (e.g. A or B), with regard to any other point of CD.



Check at G:

$$H^2 = h_1^2 + \frac{Q}{\pi K} \ln \frac{\sqrt{25^2 + 20^2}}{\sqrt{199^2 + 20^2}} \Rightarrow$$

$$\Rightarrow Q = 0.0435 \quad m^3/s$$

Check at A:

$$H^2 = h_1^2 + \frac{Q}{\pi K} \ln \frac{\sqrt{50^2 + 20^2}}{\sqrt{124^2 + 20^2}} \Rightarrow \pi K(86^2 - 88^2) = Q \ln(0.3075) \Rightarrow Q = 0.039 \quad m^3/s$$

Exercise 3. In a semi-infinite aquifer, bound by a lake, there are 5 wells (A, B, C, D, and E), as shown in the figure. Two of them will be used, in order to pump ground water at a rate of $Q = 0.02\ m^3/s$ each.

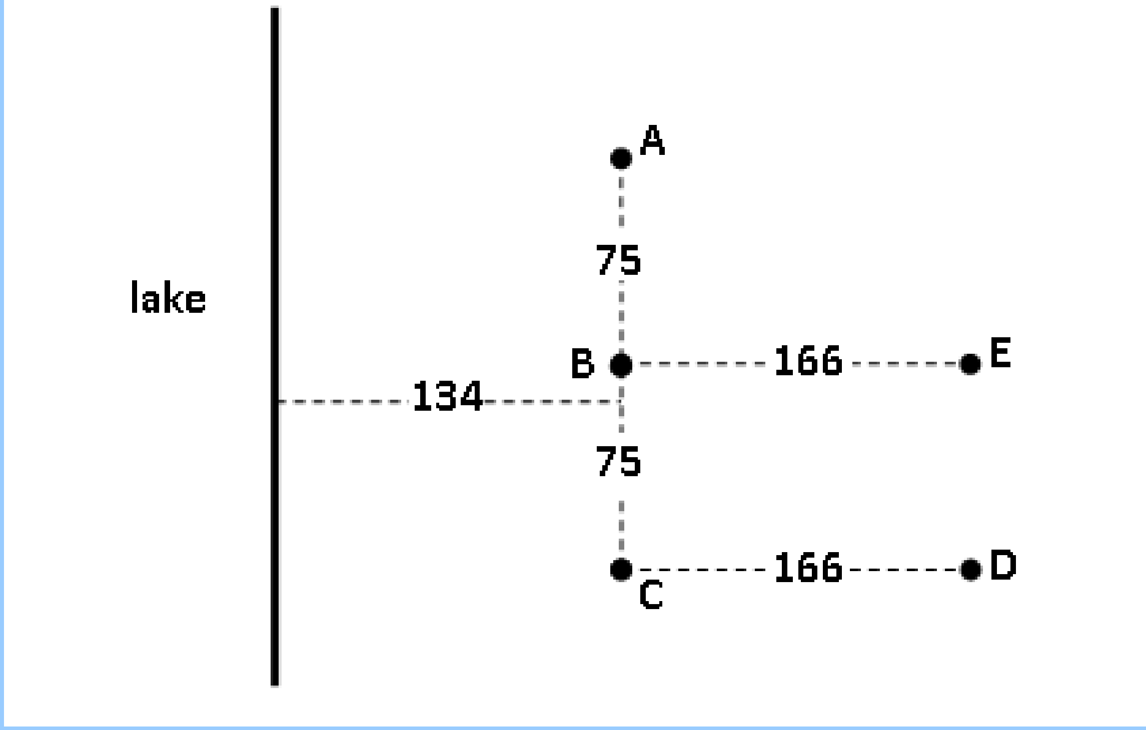
Which wells should be used, in order to minimize the pumping cost? Calculate that cost (as a function of coefficient C1).

Data: a) Aquifer's hydraulic conductivity is $K = 0.00002$ m/s b) Aquifer's width is $a = 54$ m, c) The radius of all wells is $r_0 = 0.2$ m, d) The flow is confined everywhere e) Pumping cost PC is given as:

$$PC = C1(Q_1 s_1 + Q_2 s_2)$$

where $Q_1 = Q_2 = Q$ are the flow rates of the two pumping wells and s_1, s_2 the respective hydraulic head level drawdowns at those wells.

Explain the choice of the pumping wells.

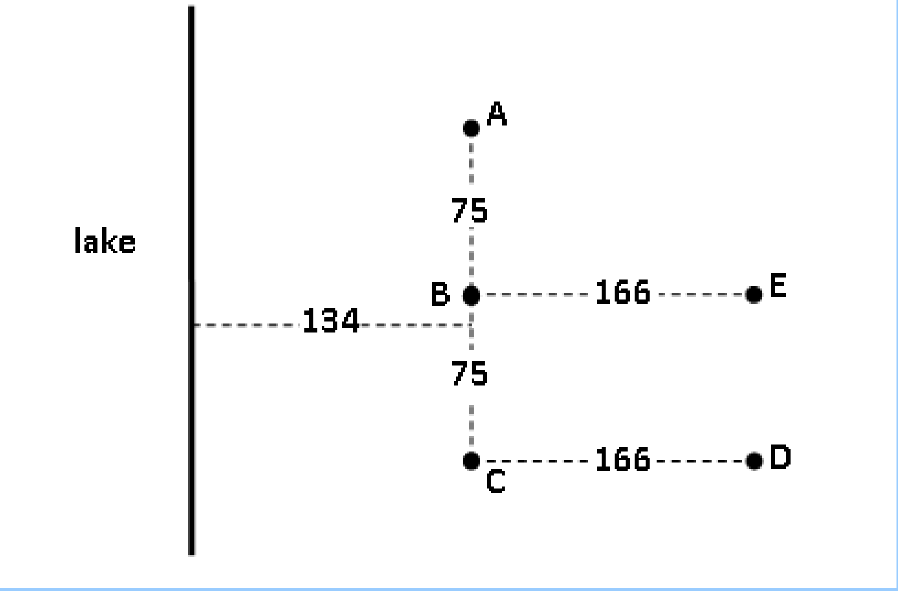


Solution

The cost is minimized when s is minimized.

For a single well, pumping a given Q, s becomes smaller with the distance between the lake and the wells

For systems of wells, s becomes smaller when the distance between wells becomes larger.



So, we have to check 2 combinations of wells:

a) **Wells A and C and**

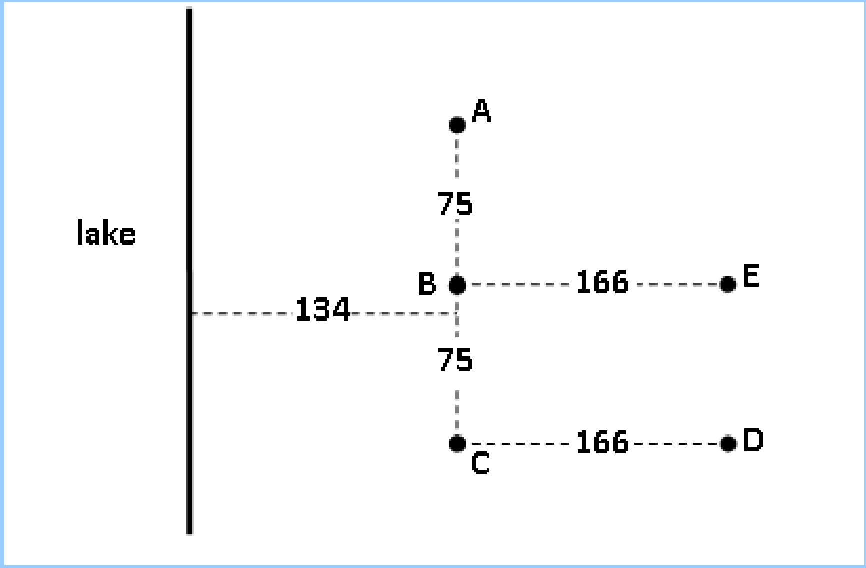
b) **Wells A and D.**

According to the method of images, s is given as:

$$s = -\frac{1}{2\pi K a}\sum_{i=1}^{n} Q_i \cdot \ln \frac{\sqrt{(x-x_i)^2 + (y-y_i)^2}}{\sqrt{(x+x_i)^2 + (y-y_i)^2}}$$

Wells A and C

$$s_A = -\frac{Q}{2\pi Ka}\ln\frac{0.2}{268} - \frac{Q}{2\pi Ka}\ln\frac{150}{\sqrt{268^2+150^2}} = 21.24 + 2.11 = 23.35\ \text{m}$$



$s_C = s_A$ =23.35 m, due to symmetry.

Then PC = C1(2·0.02·23.35) = 0.934 C1

Wells A and D

$$s_A = -\frac{Q}{2\pi Ka}\ln\frac{0.2}{268} - \frac{Q}{2\pi Ka}\ln\frac{\sqrt{166^2+150^2}}{\sqrt{434^2+150^2}} = 21.24 + 2.12 = 23.36\ \text{m}$$

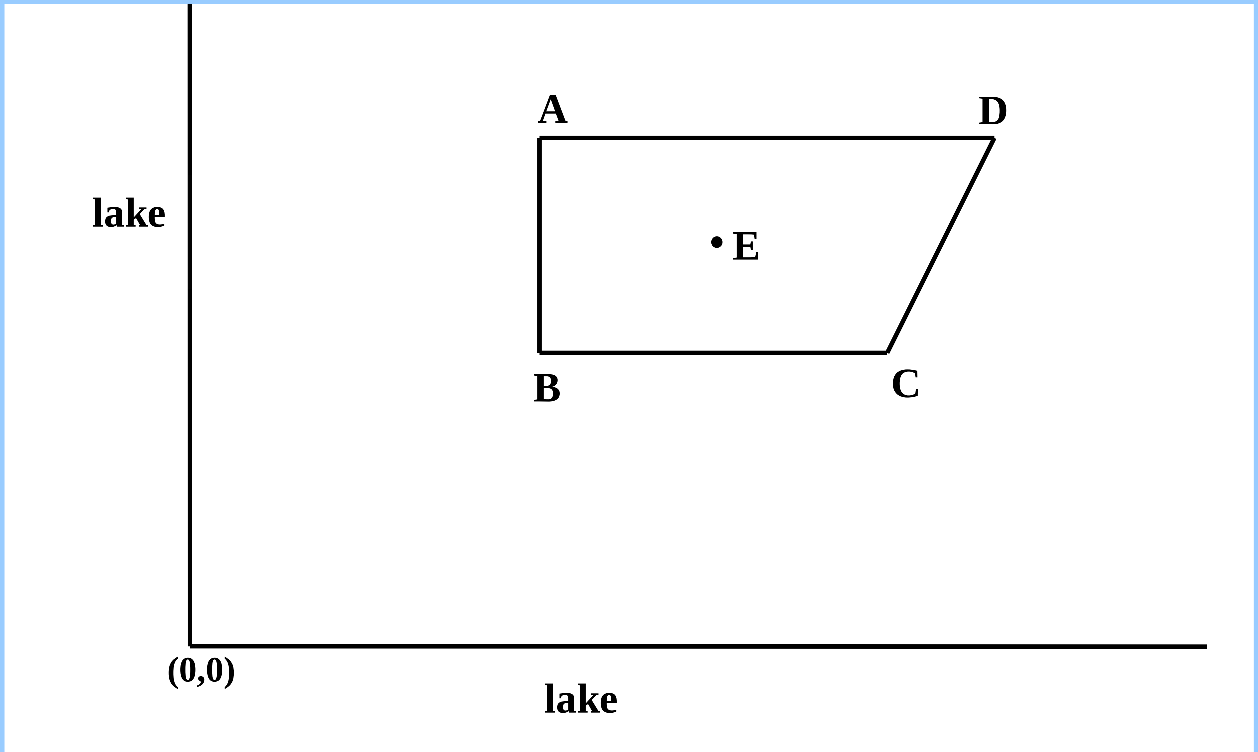
$$s_D = -\frac{Q}{2\pi Ka}\ln\frac{\sqrt{166^2+150^2}}{\sqrt{434^2+150^2}} - \frac{Q}{2\pi Ka}\ln\frac{0.2}{600} = 2.12 + 23.62 = 25.74\ \text{m}$$

Then PC = C1(0.02·23.36 + 0.02·25.74) = 0.982 C1

Exercise 4

Well E pumps groundwater at a rate of Q_E = 44 *l/s* from the underlying semi-infinite aquifer, which is bound by a lake. The owner of plot ABCD will construct a second well, in order to pump an additional flow rate of 32 *l/s*. At which point should he construct the new well, in order that the hydraulic head level drawdown at that new well be minimal? Calculate that hydraulic head level drawdown value.

Data: a) Aquifer width a = 50 m and Hydraulic conductivity K= 10^{-4} m/s b) Radius of each well r_0= 0.20 m c) The flow is steady and confined. d) The coordinates of A, B, C, D and E are: A(120,160), B(120, 110), C(170,110), D(210,160) and E(145,135).



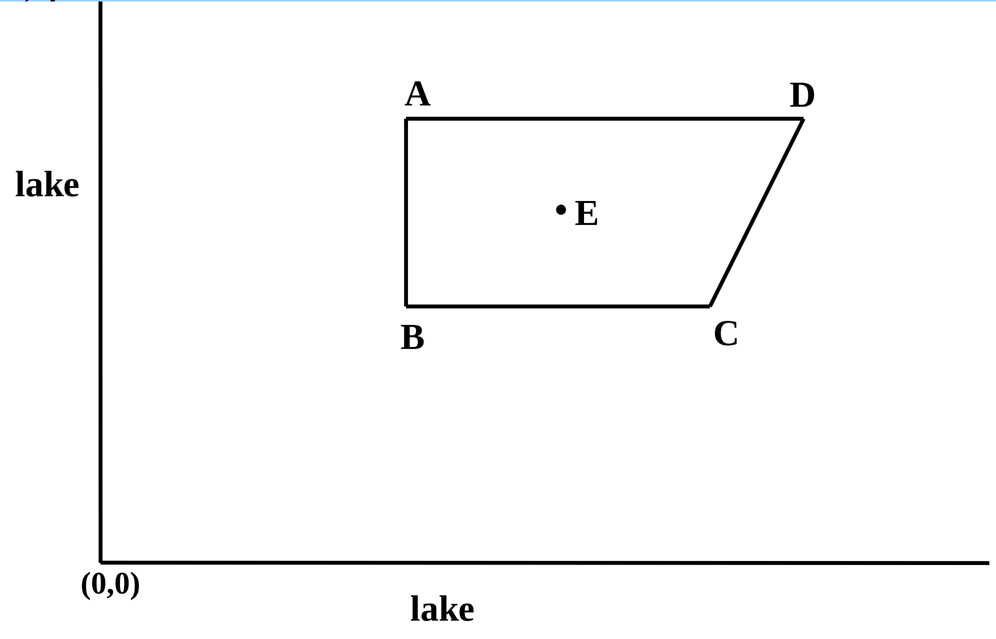
The hydraulic head level drawdown at any point (x,y) is given as:

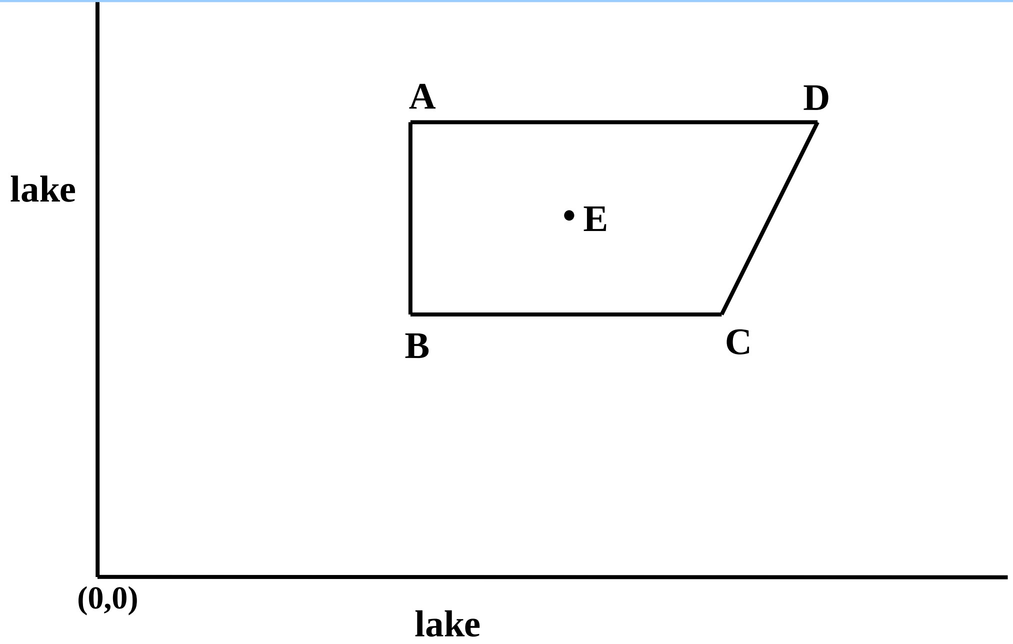
$$s = -\frac{1}{2\pi K a}\sum_{i=1}^{n} Q_i \ln \frac{\sqrt{(x-x_i)^2+(y-y_i)^2}\sqrt{(x+x_i)^2+(y+y_i)^2}}{\sqrt{(x-x_i)^2+(y+y_i)^2}\sqrt{(x+x_i)^2+(y-y_i)^2}}$$

If the new well is constructed at B, the hydraulic head level drawdown will be:

$$s_B = -\frac{1}{2\pi K a}\left(Q_B \ln\frac{0.2\sqrt{240^2+220^2}}{220\cdot 240} + Q_E \ln\frac{\sqrt{25^2+25^2}\sqrt{(120+145)^2+(110+135)^2}}{\sqrt{25^2+245^2}\sqrt{25^2+265^2}}\right) \Rightarrow$$

$$s_B = -31.847\left[0.032\left(-6.7\right)+0.044(-1.636)\right] = 9.12 \text{ m}$$





If the new well is constructed at D, the hydraulic head level drawdown will be:

$$s_D = -\frac{1}{2\pi Ka}\left(Q_D \ln\frac{0.2\sqrt{420^2+320^2}}{420\cdot 320} + Q_E \ln\frac{\sqrt{65^2+25^2}\sqrt{(210+145)^2+(160+135)^2}}{\sqrt{65^2+(135+160)^2}\sqrt{(145+210)^2+25^2}}\right) \Rightarrow$$

$$s_D = -31.847\left[0.032\left(-7.149\right)+0.044\left(-1.207\right)\right] = 8.97\ \text{m}$$

Exercise 5.

What is the maximum well flow rate Q_{max} that can be pumped through well A from the semi-infinite aquifer that is shown in the figure, without violating the following constraints: a) The hydraulic head level drawdown should not exceed 31 m at any point of the aquifer and b) The hydraulic head level drawdown should not exceed 17 m at any point of the protected area.

Data:

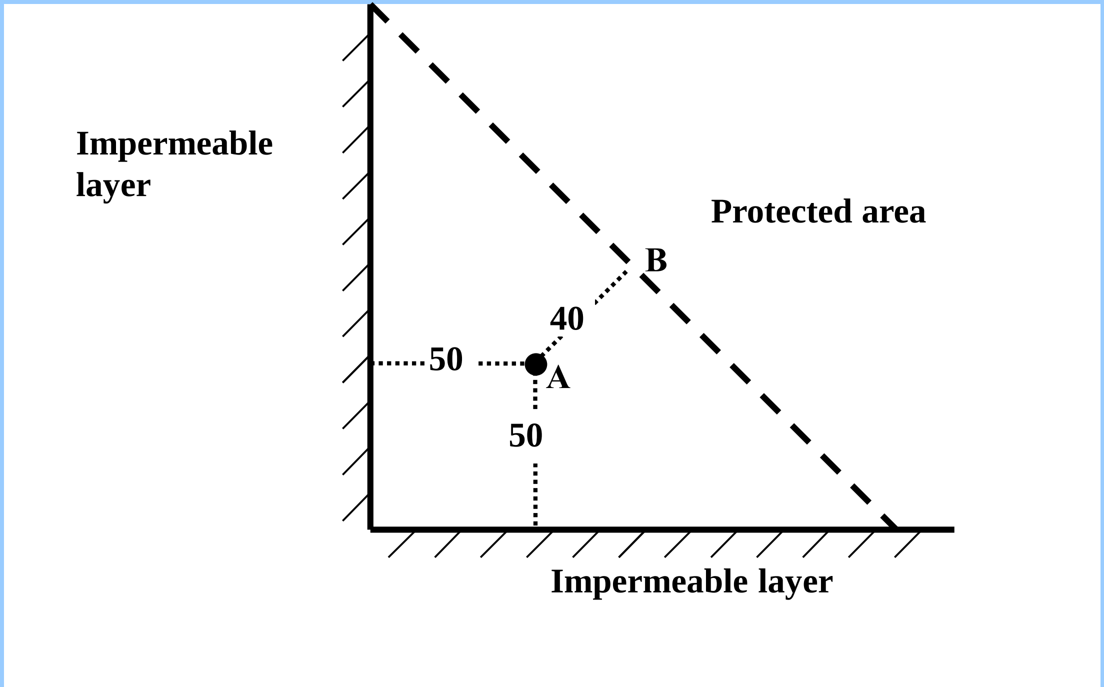
a) Aquifer width a = 50 m

b) Hydraulic conductivity K= 10^{-4} m/s

c) Well radius r_0= 0.20 m

d) Radius of influence R = 3000 m

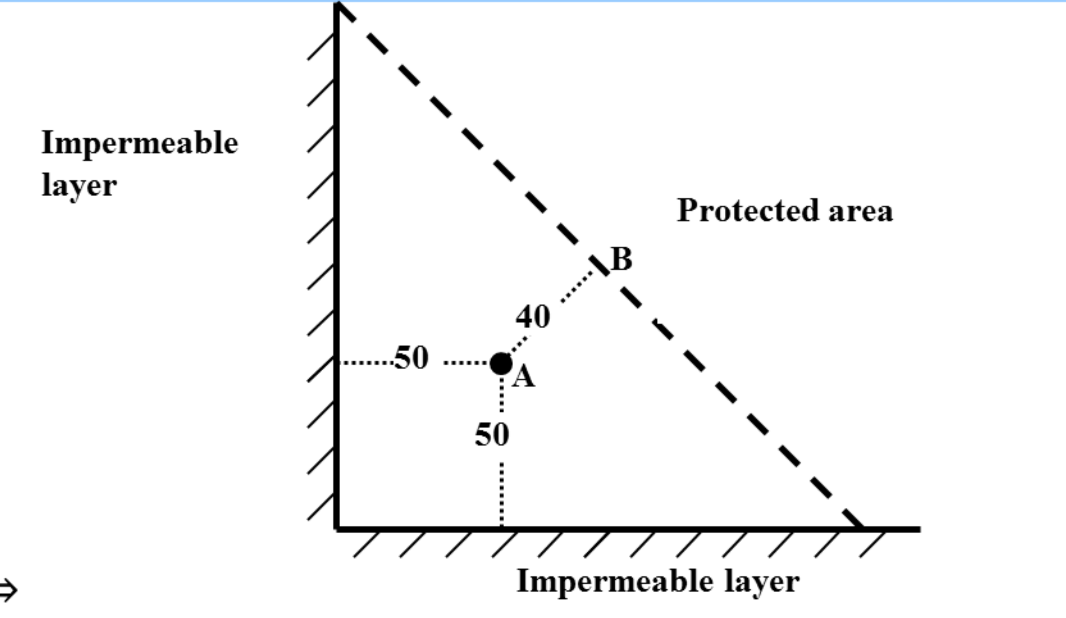
e) Flow is confined and steady.



Hint: Point B of the protected area, which is the closest to the well, should be checked with regard to the second constraint.

Solution

According to the method of images, 3 imaginary wells should be used.



Hydraulic head level drawdown at the location of the well is given as:

$$s_A = -\frac{Q}{2\pi Ka} ln \frac{0.2 \cdot 100 \cdot 100 \cdot 100\sqrt{2}}{R^4} \Rightarrow$$

$$\Rightarrow 31 = -\frac{Q}{2\pi Ka}(-19.473) \Rightarrow Q = 0.05\frac{m^3}{s}$$

Hydraulic head level drawdown at point B is given as:

$$s_B = 17 = -\frac{Q}{2\pi Ka}\left(\ln\frac{40}{R} + 2ln\frac{\sqrt{\left(\frac{40}{\sqrt{2}}\right)^2 + \left(100 + \frac{40}{\sqrt{2}}\right)^2}}{R} + \ln\frac{100\sqrt{2} + 40}{R}\right) \Rightarrow Q = 0.04\frac{m^3}{s}$$

Licensing note

End of Unit

Thessaloniki,

December 2016